

# REAL SUBMANIFOLDS OF MAXIMUM COMPLEX TANGENT SPACE AT A CR SINGULAR POINT, II

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**ABSTRACT.** We study a germ of real analytic  $n$ -dimensional submanifold of  $\mathbf{C}^n$  that has a complex tangent space of maximal dimension at a CR singularity. Under the condition that its complexification admits the maximum number of deck transformations, we first classify holomorphically its quadratic CR singularity. We then study its transformation to a normal form under the action of local (possibly formal) biholomorphisms at the singularity. We first conjugate formally its associated reversible map  $\sigma$  to suitable normal forms and show that all these normal forms can be divergent. We then construct a unique formal normal form under a non degeneracy condition.

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## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Introduction.** We say that a point  $x_0$  in a real submanifold  $M$  in  $\mathbf{C}^n$  is a CR singularity, if the complex tangent spaces  $T_x M \cap J_x T_x M$  do not have a constant dimension in any neighborhood of  $x_0$ . The study of real submanifolds with CR singularities was initiated by E. Bishop in his pioneering work [Bis65], when the complex tangent space of  $M$  at a CR singularity is minimal, that is exactly one-dimensional. The very elementary models of this kind of manifolds are classified as the Bishop quadrics in  $\mathbf{C}^2$ , given by

$$(1.1) \quad Q: z_2 = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2), \quad 0 \leq \gamma < \infty; \quad Q: z_2 = z_1^2 + \bar{z}_1^2, \quad \gamma = \infty$$

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with Bishop invariant  $\gamma$ . The origin is a complex tangent which is said to be *elliptic* if  $0 \leq \gamma < 1/2$ , *parabolic* if  $\gamma = 1/2$ , or *hyperbolic* if  $\gamma > 1/2$ .

In [MW83], Moser and Webster studied the normal form problem of a real analytic surface  $M$  in  $\mathbf{C}^2$  which is the higher order perturbation of  $Q$ . They showed that when  $0 < \gamma < 1/2$ ,  $M$  is holomorphically equivalent to a normal form which is an algebraic surface that depends only on  $\gamma$  and two discrete invariants. They also constructed a formal normal form of  $M$  when the origin is a non-exceptional hyperbolic complex tangent point; although the normal form is still convergent, they showed that the normalization is divergent in general for the hyperbolic case. In fact, Moser-Webster dealt with an  $n$ -dimensional real submanifold  $M$  in  $\mathbf{C}^n$ , of which the complex tangent space has (minimum) dimension 1 at a CR singularity. When  $n > 2$ , they also found normal forms under suitable non-degeneracy condition.

In this paper we continue our previous investigation on an  $n$ -dimensional real analytic submanifold  $M$  in  $\mathbf{C}^n$  of which the complex tangent space has the *largest* possible dimension at a given CR singularity [GS15]. The dimension must be  $p = n/2$ . Therefore,  $n = 2p$  is even. As shown in [Sto07] and [GS15], there is yet another basic quadratic model

$$(1.2) \quad Q_{\gamma_s} \subset \mathbf{C}^4: z_3 = (z_1 + 2\gamma_s \bar{z}_2)^2, \quad z_4 = (z_2 + 2(1 - \bar{\gamma}_s)\bar{z}_1)^2$$

with  $\gamma_s$  an invariant satisfying  $\operatorname{Re} \gamma_s \leq 1/2$ ,  $\operatorname{Im} \gamma_s \geq 0$ , and  $\gamma_s \neq 0$ . The complex tangent at the origin is said of complex type. In [GS15], we obtained convergence of normalization for abelian CR singularity. In this paper, we study systematically the normal forms of the manifolds  $M$  under the condition that  $M$  admit the maximum number of deck transformations, condition D, introduced in [GS15].

In suitable holomorphic coordinates, a  $2p$ -dimensional real analytic submanifold in  $\mathbf{C}^{2p}$  that has a complex tangent space of maximum dimension at the origin is given by

$$\begin{aligned} M: z_{p+j} &= E_j(z', \bar{z}'), \quad 1 \leq j \leq p, \\ E_j(z', \bar{z}') &= h_j(z', \bar{z}') + q_j(\bar{z}') + O(|(z', \bar{z}')|^3), \end{aligned}$$

where  $z' = (z_1, \dots, z_p)$ , each  $h_j(z', \bar{z}')$  is a homogeneous quadratic polynomial in  $z', \bar{z}'$  without holomorphic or anti-holomorphic terms, and each  $q_j(\bar{z}')$  is a homogeneous quadratic polynomial in  $\bar{z}'$ . We call  $M$  a *quadratic manifold* in  $\mathbf{C}^{2p}$  if  $E_j$  are homogeneous quadratic polynomials. If  $M$  is a product of Bishop quadrics (1.1) and quadrics of the form (1.2), it is called a *product quadric*.

**1.2. Basic invariants.** We first describe some basic invariants of real analytic submanifolds, which are essential to the normal forms. To study  $M$ , we consider its complexification in  $\mathbf{C}^{2p} \times \mathbf{C}^{2p}$  defined by

$$\mathcal{M}: \begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

It is a complex submanifold of complex dimension  $2p$  with coordinates  $(z', w') \in \mathbf{C}^{2p}$ . Let  $\pi_1, \pi_2$  be the restrictions of the projections  $(z, w) \rightarrow z$  and  $(z, w) \rightarrow w$  to  $\mathcal{M}$ , respectively. Note that  $\pi_2 = C\pi_1\rho_0$ , where  $\rho_0$  is the restriction to  $\mathcal{M}$  of the anti-holomorphic involution  $(z, w) \rightarrow (\bar{w}, \bar{z})$  and  $C$  is the complex conjugate. It is proved in [GS15] that when  $M$  satisfies *condition B*, i.e.  $q^{-1}(0) = 0$ , the deck transformations of  $\pi_1$  are involutions that

commute pairwise, while the number of deck transformations can be  $2^\ell$  for  $1 \leq \ell \leq p$ . As in [GS15], our basic hypothesis on  $M$  is *condition D* that  $\pi_1$  admits the maximum number,  $2^p$ , deck transformations. Then it is proved in [GS15] that the group of deck transformations of  $\pi_1$  is generated uniquely by  $p$  involutions  $\tau_{11}, \dots, \tau_{1p}$  such that each  $\tau_{1j}$  fixes a hypersurface in  $\mathcal{M}$ . Furthermore,

$$\tau_1 := \tau_{11} \dots \tau_{1p}$$

is the unique deck transformation of which the set of the fixed-points has the smallest dimension  $p$ . We call  $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$  the set of *Moser-Webster involutions*. Let  $\tau_2 = \rho_0 \tau_1 \rho_0$  and

$$\sigma = \tau_1 \tau_2.$$

Then  $\sigma$  is *reversible* by  $\tau_j$  and  $\rho_0$ , i.e.  $\sigma^{-1} = \tau_j \sigma \tau_j^{-1}$  and  $\sigma^{-1} = \rho_0 \sigma \rho_0$ .

In this paper for classification purposes, we will impose the following condition:

**Condition E.**  *$M$  has distinct eigenvalues, i.e.  $\sigma$  has  $2p$  distinct eigenvalues.*

We now introduce our main results.

Our first step is to normalize  $\{\tau_1, \tau_2, \rho_0\}$ . When  $p = 1$ , this normalization is the main step in order to obtain the Moser-Webster normal form; in fact a simple further normalization allows Moser and Webster to achieve a convergent normal form under a suitable non-resonance condition even for the non-exceptional hyperbolic complex tangent.

When  $p > 1$ , we need to carry out a further normalization for  $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$ ; this is our second step. Here the normalization has a large degree of freedom as shown by our formal and convergence results.

**1.3. A normal form of quadrics.** In section 3, we study all quadrics which admit the maximum number of deck transformations. For such quadrics, all deck transformations are linear. Under condition E, we will first normalize  $\sigma, \tau_1, \tau_2$  and  $\rho_0$  into  $\hat{S}, \hat{T}_1, \hat{T}_2$  and  $\rho$  where

$$\begin{aligned} \hat{T}_1: \quad \xi'_j &= \lambda_j^{-1} \eta_j, & \eta'_j &= \lambda_j \xi_j, \\ \hat{T}_2: \quad \xi'_j &= \lambda_j \eta_j, & \eta'_j &= \lambda_j^{-1} \xi_j, \\ \hat{S}: \quad \xi'_j &= \mu_j \xi_j, & \eta'_j &= \mu_j^{-1} \eta_j \end{aligned}$$

with

$$\lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \overline{\lambda_s}^{-1}, \quad \mu_j = \lambda_j^2.$$

Here  $1 \leq j \leq p$ . Throughout the paper, the indices  $e, h, s$  have the ranges:  $1 \leq e \leq e_*$ ,  $e_* < h \leq e_* + h_*$ ,  $e_* + h_* < s \leq p - s_*$ . Thus  $e_* + h_* + 2s_* = p$ . We will call  $e_*, h_*, s_*$  the numbers of *elliptic*, *hyperbolic* and *complex* components of a product quadric, respectively. As in the Moser-Webster theory, at the complex tangent (the origin) an *elliptic* component of a product quadric corresponds a *hyperbolic* component of  $\hat{S}$ , while a *hyperbolic* component of the quadric corresponds an *elliptic* component of  $\hat{S}$ . On the other hand, a *complex* component of the quadric behaves like an elliptic component when the CR singularity is abelian, and it also behaves like a hyperbolic components for the existence of attached complex manifolds; see [GS15] for details.

For the above normal form of  $\hat{T}_1, \hat{T}_2$  and  $\hat{S}$ , we always normalize the anti-holomorphic involution  $\rho_0$  as

$$(1.3) \quad \rho: \begin{cases} \xi'_e &= \bar{\eta}_e, & \eta'_e &= \bar{\xi}_e, \\ \xi'_h &= \bar{\xi}_h, & \eta'_h &= \bar{\eta}_h, \\ \xi'_s &= \bar{\xi}_{s+s_*}, & \eta'_s &= \bar{\eta}_{s+s_*}, \\ \xi'_{s+s_*} &= \bar{\xi}_s, & \eta'_{s+s_*} &= \bar{\eta}_s. \end{cases}$$

With the above normal forms  $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$  with  $\hat{S} = \hat{T}_1 \hat{T}_2$ , we will then normalize the  $\tau_{11}, \dots, \tau_{1p}$  under linear transformations that commute with  $\hat{T}_1, \hat{T}_2$ , and  $\rho$ , i.e. the linear transformations belonging to the *centralizer* of  $\hat{T}_1, \hat{T}_2$  and  $\rho$ . This is a subtle step. Instead of normalizing the involutions directly, we will use the pairwise commutativity of  $\tau_{11}, \dots, \tau_{1p}$  to associate to these  $p$  involutions a non-singular  $p \times p$  matrix  $\mathbf{B}$ . The normalization of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  is then identified with the normalization of the matrices  $\mathbf{B}$  under a suitable equivalence relation. The latter is easy to solve. Our normal form of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  is then constructed from the normal forms of  $T_1, T_2, \rho$ , and the matrix  $\mathbf{B}$ . Following Moser-Webster [MW83], we will construct the normal form of the quadrics from the normal form of involutions. Let us first state a Bishop type holomorphic classification for quadratic real manifolds.

**Theorem 1.1.** *Let  $M$  be a quadratic submanifold defined by*

$$z_{p+j} = h_j(z', \bar{z}') + q_j(\bar{z}'), \quad 1 \leq j \leq p.$$

*Suppose that  $M$  satisfies condition E, i.e. the branched covering of  $\pi_1$  of complexification  $\mathcal{M}$  has  $2^p$  deck transformations and  $2p$  distinct eigenvalues. Then  $M$  is holomorphically equivalent to*

$$Q_{\mathbf{B}, \gamma}: z_{p+j} = L_j^2(z', \bar{z}'), \quad 1 \leq j \leq p$$

*where  $(L_1(z', \bar{z}'), \dots, L_p(z', \bar{z}'))^t = \mathbf{B}(z' - 2\gamma\bar{z}')$ ,  $\mathbf{B} \in GL_p(\mathbf{C})$  and*

$$\gamma := \begin{pmatrix} \gamma_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} - \bar{\gamma}_{s_*} & \mathbf{0} \end{pmatrix}.$$

*Here  $p = e_* + h_* + 2s_*$ ,  $\mathbf{I}_{s_*}$  denotes the  $s_* \times s_*$  identity matrix, and*

$$\begin{aligned} \gamma_{e_*} &= \text{diag}(\gamma_1, \dots, \gamma_{e_*}), \quad \gamma_{h_*} = \text{diag}(\gamma_{e_*+1}, \dots, \gamma_{e_*+h_*}), \\ \gamma_{s_*} &= \text{diag}(\gamma_{e_*+h_*+1}, \dots, \gamma_{p-s_*}) \end{aligned}$$

*with  $\gamma_e, \gamma_h$ , and  $\gamma_s$  satisfying*

$$0 < \gamma_e < 1/2, \quad 1/2 < \gamma_h < \infty, \quad \text{Re } \gamma_s < 1/2, \quad \text{Im } \gamma_s > 0.$$

*Moreover,  $\mathbf{B}$  is uniquely determined by an equivalence relation  $\mathbf{B} \sim \mathbf{CBR}$  for suitable non-singular matrices  $\mathbf{C}, \mathbf{R}$  which have exactly  $p$  non-zero entries.*

When  $\mathbf{B}$  is the identity matrix, we get a product quadric or its equivalent form. See Theorem 3.7 for detail of the equivalence relation. The scheme of finding quadratic normal forms turns out to be useful. It will be applied to the study of normal forms of the general real submanifolds.

**1.4. Formal submanifolds, formal involutions, and formal centralizers.** The normal forms of  $\sigma$  turn out to be in the centralizer of  $\hat{S}$ , the normal form of the linear part of  $\sigma$ . The family is subject to a second step of normalization under mappings which again turn out to be in the centralizer of  $\hat{S}$ . Thus, before we introduce normalization, we will first study various centralizers. We will discuss the centralizer of  $\hat{S}$  as well as the centralizer of  $\{\hat{T}_1, \hat{T}_2\}$  in section 4.

**1.5. Normalization of  $\sigma$ .** As mentioned earlier, we will divide the normalization for the families of non-linear involutions into two steps. This division will serve two purposes: first, it helps us to find the formal normal forms of the family of involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ ; second, it helps us understand the convergence of normalization of the original normal form problem for the real submanifolds. For purpose of normalization, we will assume that  $M$  is *non-resonant*, i.e.  $\sigma$  is *non-resonant*, if its eigenvalues  $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$  satisfy

$$(1.4) \quad \mu^Q \neq 1, \quad \forall Q \in \mathbf{Z}^p, \quad |Q| \neq 0.$$

In section 5, we obtain the normalization of  $\sigma$  by proving the following.

**Theorem 1.2.** *Let  $\sigma$  be a holomorphic map with linear part  $\hat{S}$ . Assume that  $\hat{S}$  has eigenvalues  $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$  satisfying the non-resonant condition (1.4). Suppose that  $\sigma = \tau_1 \tau_2$  where  $\tau_1$  is a holomorphic involution,  $\rho$  is an anti-holomorphic involution, and  $\tau_2 = \rho \tau_1 \rho$ . Then there exists a formal map  $\Psi$  such that  $\rho := \Psi^{-1} \rho \Psi$  is given by (1.3),  $\sigma^* = \Psi^{-1} \sigma \Psi$  and  $\tau_i^* = \Psi^{-1} \tau_i \Psi$  have the form*

$$(1.5) \quad \begin{aligned} \sigma^*: \xi'_j &= M_j(\xi\eta)\xi_j, & \eta'_j &= M_j^{-1}(\xi\eta)\eta_j, & M_j(0) &= \mu_j, & 1 \leq j \leq p, \\ \tau_i^* &= \Lambda_{ij}(\xi\eta)\eta_j, & \eta'_j &= \Lambda_{ij}^{-1}(\xi\eta)\xi_j. \end{aligned}$$

Here,  $\xi\eta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$ . Assume further that  $\log M$  (see (5.33) for definition) is tangent to the identity. Under a further change of coordinates that preserves  $\rho$ ,  $\sigma^*$  and  $\tau_i^*$  are transformed into

$$(1.6) \quad \begin{aligned} \hat{\sigma}: \xi'_j &= \hat{M}_j(\xi\eta)\xi_j, & \eta'_j &= \hat{M}_j^{-1}(\xi\eta)\eta_j, & \hat{M}_j(0) &= \mu_j, & 1 \leq j \leq p, \\ \hat{\tau}_i &= \hat{\Lambda}_{ij}(\xi\eta)\eta_j, & \eta'_j &= \hat{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j, & \hat{\Lambda}_{2j} &= \hat{\Lambda}_{1j}^{-1}. \end{aligned}$$

Here the  $j$ th component of  $\log \hat{M}(\zeta) - \zeta = O(|\zeta|^2)$  is independent of  $\zeta_j$ . Moreover,  $\hat{M}$  is unique.

**Remark 1.3.** The condition that  $\log M$  is tangent to identity at the origin has to be understood as a non-degeneracy condition of the simplest form. When there is no ambiguity, “tangent to identity” stands for “tangent to identity at the origin”.

We will conclude section 5 with an example showing that although  $\sigma, \tau_1, \tau_2$  are linear,  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  are not necessarily linearizable, provided  $p > 1$ .

Section 6 is devoted to the proof of the following divergence result.

**Theorem 1.4.** *There exists a non-resonant real analytic submanifold  $M$  with pure elliptic complex tangent in  $\mathbf{C}^6$  such that if its associated  $\sigma$  is transformed into a map  $\sigma^*$  that commutes with the linear part of  $\sigma$  at the origin, then  $\sigma^*$  must diverge.*

Note that the theorem says that all normal forms of  $\sigma$  (by definition, they belong to the centralizer of its linear part, i.e. they are in the Poincaré-Dulac normal forms) are divergent. It implies that any transformation for  $M$  that transforms  $\sigma$  into a Poincaré-Dulac normal form must diverge. This is in contrast with the Moser-Webster theory: For  $p = 1$ , a convergent normal form can always be achieved even if the associated transformation is divergent (in the case of hyperbolic complex tangent), and furthermore in case of  $p = 1$  and elliptic complex tangent with a non-vanishing Bishop invariant, the normal form can be achieved by a convergent transformation. A divergent Birkhoff normal form for the classical Hamiltonian systems was obtained in [Go12]. See Yin [Yi15] for the existence of divergent Birkhoff normal forms for real analytic area-preserving mappings.

We do not know if there exists a non-resonant real analytic submanifold with pure elliptic eigenvalues in  $\mathbf{C}^4$  of which all Poincaré-Dulac normal forms are divergent.

**1.6. A unique normalization for the family  $\{\tau_{ij}, \rho\}$ .** In section 7, we will follow the normalization scheme developed for the quadric normal forms in order to normalize  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ . Let  $\hat{\sigma}$  be given by (1.6). We define

$$\hat{\tau}_{1j}: \xi'_j = \hat{\Lambda}_{1j}(\xi\eta)\eta_j, \quad \eta'_j = \hat{\Lambda}_{1j}^{-1}(\xi\eta)\xi_j, \quad \xi'_k = \xi_k, \quad \eta'_k = \eta_k, \quad k \neq j,$$

where  $\hat{\Lambda}_{1j}(0) = \lambda_j$  and  $\hat{M}_j = \hat{\Lambda}_{1j}^2$ . We have the following formal normal form.

**Theorem 1.5.** *Let  $M$  be a real analytic submanifold that is a higher order perturbation of a non-resonant product quadric. Suppose that its associated  $\sigma$  is formally equivalent to  $\hat{\sigma}$  given by (1.6). Suppose that the formal mapping  $\log \hat{M}$  is as in Theorem 1.2. Then the formal normal form of  $M$  is completely determined by*

$$\hat{M}(\zeta), \quad \Phi(\xi, \eta).$$

Here the formal mapping  $\Phi$  is in  $\mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}) \cap \mathcal{C}(\hat{\tau}_1)$  and tangent to the identity. Moreover,  $\Phi$  is uniquely determined up to the equivalence relation  $\Phi \sim R_\epsilon \Phi R_\epsilon^{-1}$  with  $R_\epsilon: \xi_j = \epsilon_j \xi, \eta'_j = \epsilon_j \eta_j$  ( $1 \leq j \leq p$ ),  $\epsilon_j^2 = 1$  and  $\epsilon_{s+s_*} = \epsilon_s$ . Furthermore, if the normal form (1.5) of  $\sigma$  can be achieved by a convergent transformation, so does the normal form of  $M$ .

The set  $\mathcal{C}(\hat{\tau}_1) \cap \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$  is given in Lemma 7.2 with  $\mathbf{B}_1$  being the identity matrix.

We now mention related normal form problems. The normal form problem, that is the equivalence to a model manifold, of analytic real hypersurfaces in  $\mathbf{C}^n$  with a non-degenerate Levi-form has a complete theory achieved through the works of E. Cartan [Car32], [Car33], Tanaka [Tan62], and Chern-Moser [CM74]. In another direction, the relations between formal and holomorphic equivalences of real analytic hypersurfaces (thus there is no CR singularity) have been investigated by Baouendi-Ebenfelt-Rothschild [BER97], [BER00], Baouendi-Mir-Rothschild [BMR02], and Juhlin-Lamel [JL13], where positive (i.e. convergent) results were obtained. In a recent paper, Kossovskiy and Shafikov [KS13] showed that there are real analytic real hypersurfaces which are formally but not holomorphically equivalent. In the presence of CR singularity, the problems and techniques required are however different from those used in the CR case. See [GS15] for further references and therein.

**1.7. Notation.** We briefly introduce notation used in the paper. The identity map is denoted by  $I$ . The matrix of a linear map  $y = Ax$  is denoted by a bold-faced  $\mathbf{A}$ . We denote by  $LF$  the linear part at the origin of a mapping  $F: \mathbf{C}^m \rightarrow \mathbf{C}^n$  with  $F(0) = 0$ . Let  $F'(0)$  or  $DF(0)$  denote the Jacobian matrix of the  $F$  at the origin. Then  $LF(z) = F'(0)z$ . We also denote by  $DF(z)$  or simply  $DF$ , the Jacobian matrix of  $F$  at  $z$ , when there is no ambiguity. If  $\mathcal{F}$  is a family of mappings fixing the origin, let  $L\mathcal{F}$  denote the family of linear parts of mappings in  $\mathcal{F}$ . By an analytic (or holomorphic) function, we shall mean a *germ* of analytic function at a point (which will be defined by the context) otherwise stated. We shall denote by  $\mathcal{O}_n$  (resp.  $\widehat{\mathcal{O}}_n$ ,  $\mathfrak{M}_n$ ,  $\widehat{\mathfrak{M}}_n$ ) the space of germs of holomorphic functions of  $\mathbf{C}^n$  at the origin (resp. of formal power series in  $\mathbf{C}^n$ , holomorphic germs, and formal germs vanishing at the origin).

## 2. MOSER-WEBSTER INVOLUTIONS AND PRODUCT QUADRICS

In this section we will first recall a formal and convergent result from [GS15] that will be used to classify real submanifolds admitting the maximum number of deck transformations. We will then derive the family of deck transformations for the product quadrics.

We consider a formal real submanifold of dimension  $2p$  in  $\mathbf{C}^{2p}$  defined by

$$(2.1) \quad M: z_{p+j} = E_j(z', \bar{z}'), \quad 1 \leq j \leq p.$$

Here  $E_j$  are formal power series in  $z', \bar{z}'$ . We assume that

$$(2.2) \quad E_j(z', \bar{z}') = h_j(z', \bar{z}') + q_j(\bar{z}') + O(|(z', \bar{z}')|^3)$$

and  $h_j, q_j$  are homogeneous quadratic polynomials. The formal complexification of  $M$  is defined by

$$\mathcal{M}: \begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

We define a *formal deck transformation* of  $\pi_1$  to be a formal biholomorphic map

$$\tau: (z', w') \rightarrow (z', f(z', w')), \quad \tau(0) = 0$$

such that  $\pi_1 \tau = \pi_1$ , i.e.  $E \circ \tau = E$ . Assume that  $q^{-1}(0) = 0$  and that the formal manifold defined by (2.1)-(2.2) satisfies condition D that its formal branched covering  $\pi_1$  admits  $2^p$  formal deck transformations. Then  $\pi$  admits a unique set of  $p$  deck transformations  $\{\tau_{11}, \dots, \tau_{1p}\}$  such that each  $\tau_{1j}$  fixes a hypersurface in  $\mathcal{M}$ .

As in the Moser-Webster theory, the significance of the two sets of involutions is the following proposition that transforms the normalization of the real manifolds into that of two families  $\{\tau_{i1}, \dots, \tau_{ip}\}$  ( $i = 1, 2$ ) of commuting involutions satisfying  $\tau_{2j} = \rho \tau_{1j} \rho$  for an antiholomorphic involution  $\rho$ . Let us recall the anti-holomorphic involution

$$(2.3) \quad \rho_0: (z', w') \rightarrow (\bar{w}', \bar{z}').$$

**Proposition 2.1.** *Let  $M, \tilde{M}$  be formal (resp. real analytic) real submanifolds of dimension  $2p$  in  $\mathbf{C}^n$  of the form (2.1)-(2.2). Suppose that  $M, \tilde{M}$  satisfy condition D. Then the following hold :*

- (i)  $M$  and  $\tilde{M}$  are formally (resp. holomorphically) equivalent if and only if their associated families of involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$  and  $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho_0\}$  are formally (resp. holomorphically) equivalent.
- (ii) Let  $\mathcal{T}_1 = \{\tau_{11}, \dots, \tau_{1p}\}$  be a family of formal holomorphic (resp. holomorphic) commuting involutions such that the tangent spaces of  $\text{Fix}(\tau_{11}), \dots, \text{Fix}(\tau_{1p})$  are hyperplanes intersecting transversally at the origin. Let  $\rho$  be an anti-holomorphic formal (resp. holomorphic) involution and let  $\mathcal{T}_2 = \{\tau_{21}, \dots, \tau_{2p}\}$  with  $\tau_{2j} = \rho\tau_{1j}\rho$ . Let  $[\mathfrak{M}_n]_1^{L\mathcal{T}_i}$  be the set of linear functions without constant terms that are invariant by  $L\mathcal{T}_i$ . Suppose that

$$(2.4) \quad [\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}.$$

There exists a formal (resp. real analytic) submanifold defined by

$$(2.5) \quad z'' = (B_1^2, \dots, B_p^2)(z', \bar{z}')$$

for some formal (resp. convergent) power series  $B_1, \dots, B_p$  such that  $M$  satisfies condition D. The set of involutions  $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho_0\}$  of  $M$  is formally (resp. holomorphically) equivalent to  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ .

The above proposition is proved in [GS15, Propositions 2.8 and 3.2]. Since we need to apply the realization several times, let us recall how (2.5) is constructed. Using the fact that  $\tau_{11}, \dots, \tau_{1p}$  are commuting involutions of which the sets of fixed points are hypersurfaces intersecting transversally, we ignore  $\rho$  and linearize them simultaneously as

$$Z_j: z_{p+j} \rightarrow -z_{p+i}, \quad z_i \rightarrow z_i, \quad i \neq j$$

for  $1 \leq j \leq p$ . Thus in  $z$  coordinates, invariant functions of  $\tau_{11}, \dots, \tau_{1p}$  are generated by  $z_1, \dots, z_p$  and  $z_{p+1}^2, \dots, z_{2p}^2$ . In the original coordinates,  $z_j = A_j(\xi, \eta)$ ,  $1 \leq j \leq p$ , are invariant by the involutions, while  $z_{p+j} = \tilde{B}_j(\xi, \eta)$  is skew-invariant by  $\tau_{1j}$ . Then  $\overline{A_j(\xi, \eta)}$  are invariant by the second family  $\{\tau_{2i}\}$ . Condition (2.4) ensures that  $\varphi: (z', w') = (A(\xi, \eta), \overline{A \circ \rho(\xi, \eta)})$  is a germ of formal (biholomorphic) mapping at the origin. Then

$$M: z_{p+j} = \tilde{B}_j^2 \circ \varphi^{-1}(z', \bar{z}'), \quad 1 \leq j \leq p$$

is a realization for  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  in the sense stated in the above proposition.

Next we recall the deck transformations for a product quadric from [GS15].

Let us first recall involutions in [MW83] where the complex tangents are elliptic (with non-vanishing Bishop invariant) or hyperbolic. When  $\gamma_1 \neq 0$ , the non-trivial deck transformations of

$$Q_{\gamma_1}: z_2 = |z_1|^2 + \gamma_1(z_1^2 + \bar{z}_1^2)$$

for  $\pi_1, \pi_2$  are  $\tau_1$  and  $\tau_2$ , respectively. They are

$$\tau_1: z'_1 = z_1, \quad w'_1 = -w_1 - \gamma_1^{-1}z_1; \quad \tau_2 = \rho\tau_1\rho$$

with  $\rho$  being defined by (2.3). Here the formula is valid for  $\gamma_1 = \infty$  (i.e.  $\gamma_1^{-1} = 0$ ). Note that  $\tau_1$  and  $\tau_2$  do not commute and  $\sigma = \tau_1\tau_2$  satisfies

$$\sigma^{-1} = \tau_i\sigma\tau_i = \rho\sigma\rho, \quad \tau_i^2 = I, \quad \rho^2 = I.$$



When the complex tangent is not parabolic, the eigenvalues of  $\sigma$  are  $\mu, \mu^{-1}$  with  $\mu = \lambda^2$  and  $\gamma\lambda^2 - \lambda + \gamma = 0$ . For the elliptic complex tangent, we can choose a solution  $\lambda > 1$ , and in suitable coordinates we obtain

$$\begin{aligned}\tau_1: \xi' &= \lambda\eta + O(|(\xi, \eta)|^2), \quad \eta' = \lambda^{-1}\xi + O(|(\xi, \eta)|^2), \\ \tau_2 &= \rho\tau_1\rho, \quad \rho(\xi, \eta) = (\bar{\eta}, \bar{\xi}), \\ \sigma: \xi' &= \mu\xi + O(|(\xi, \eta)|^2), \quad \eta' = \mu^{-1}\eta + O(|(\xi, \eta)|^2), \quad \mu = \lambda^2.\end{aligned}$$

When the complex tangent is hyperbolic, i.e.  $1/2 < \gamma \leq \infty$ ,  $\tau_i$  and  $\sigma$  still have the above form, while  $|\mu| = 1 = |\lambda|$  and

$$\rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}).$$

We recall from [MW83] that

$$\gamma_1 = \frac{1}{\lambda + \lambda^{-1}}.$$

Note that for a parabolic Bishop surface, the linear part of  $\sigma$  is not diagonalizable.

Consider a quadric of the complex type of CR singularity

$$(2.6) \quad Q_{\gamma_s}: z_3 = z_1\bar{z}_2 + \gamma_s\bar{z}_2^2 + (1 - \gamma_s)z_1^2, \quad z_4 = \bar{z}_3.$$

Here  $\gamma_s$  is a complex number.

By condition B, we know that  $\gamma_s \neq 0, 1$ . Recall from [GS15] that the deck transformations for  $\pi_1$  are generated by two involutions

$$\tau_{11}: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = -w_1 - (1 - \bar{\gamma}_s)^{-1}z_2, \\ w'_2 = w_2; \end{cases} \quad \tau_{12}: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = -w_2 - \gamma_s^{-1}z_1. \end{cases}$$

We still have  $\rho$  defined by (2.3). Then  $\tau_{2j} = \rho\tau_{1j}\rho$ ,  $j = 1, 2$ , are given by

$$\tau_{21}: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = w_2; \end{cases} \quad \tau_{22}: \begin{cases} z'_1 = z_1, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = w_1, \\ w'_2 = w_2. \end{cases}$$

Thus  $\tau_i = \tau_{i1}\tau_{i2}$  is the unique deck transformation of  $\pi_i$  that has the smallest dimension of the fixed-point set among all deck transformations. They are

$$\tau_1: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = w_1, \\ w'_2 = w_2; \end{cases} \quad \tau_2: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = -w_1 - (1 - \bar{\gamma}_s)^{-1}z_2, \\ w'_2 = -w_2 - \gamma_s^{-1}z_1. \end{cases}$$

Also  $\sigma_{s1} := \tau_{11}\tau_{22}$  and  $\sigma_{s2} := \tau_{12}\tau_{21}$  are given by

$$\sigma_{s1}: \begin{cases} z'_1 = z_1, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = (1 - \bar{\gamma}_s)^{-1}z_2 + ((\bar{\gamma}_s - \bar{\gamma}_s^2)^{-1} - 1)w_1, \\ w'_2 = w_2; \end{cases}$$

$$\sigma_{s2}: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = \gamma_s^{-1}z_1 + ((\gamma_s - \gamma_s^2)^{-1} - 1)w_2. \end{cases}$$

And  $\tau_1\tau_2 = \sigma_{s1}\sigma_{s2}$  is given by

$$\sigma_s: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = (1 - \bar{\gamma}_s)^{-1}z_2 + ((\bar{\gamma}_s - \bar{\gamma}_s^2)^{-1} - 1)w_1, \\ w'_2 = \gamma_s^{-1}z_1 + ((\gamma_s - \gamma_s^2)^{-1} - 1)w_2. \end{cases}$$

Suppose that  $\gamma_s \neq 1/2$ . The eigenvalues of  $\sigma_s$  are

$$(2.7) \quad \mu_s, \quad \mu_s^{-1}, \quad \bar{\mu}_s^{-1}, \quad \bar{\mu}_s,$$

$$(2.8) \quad \mu_s = \bar{\gamma}_s^{-1} - 1.$$

Here if  $\mu_s = \bar{\mu}_s$  and  $\mu_s^{-1} = \bar{\mu}_s^{-1}$  then each eigenspace has dimension 2. Under suitable linear coordinates, the involution  $\rho$ , defined by (2.3), takes the form

$$(2.9) \quad \rho(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_2, \bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1).$$

Moreover, for  $j = 1, 2$ , we have  $\tau_{2j} = \rho\tau_{1j}\rho$  and

$$\tau_{1j}: \xi'_j = \lambda_j \eta_j, \quad \eta'_j = \lambda_j^{-1} \xi_j; \quad \xi'_i = \xi_i, \quad \eta'_i = \eta_i, \quad i \neq j;$$

$$\lambda_1 = \lambda_s, \quad \lambda_2 = \bar{\lambda}_s^{-1}, \quad \mu_s = \lambda_s^2.$$

By a permutation of coordinates that preserves  $\rho$ , we obtain a unique holomorphic invariant  $\mu_s$  satisfying

$$(2.10) \quad |\mu_s| \geq 1, \quad \text{Im } \mu_s \geq 0, \quad 0 \leq \arg \lambda_s \leq \pi/2, \quad \mu_s \neq -1.$$

By condition E, we have  $|\mu_s| \neq 1$ .

Although the case  $\gamma_s = 1/2$  is not studied in this paper, we remark that when  $\gamma_s = 1/2$  the only eigenvalue of  $\sigma_{s1}$  is 1. We can choose suitable linear coordinates such that  $\rho$  is given by (2.9), while

$$(2.11) \quad \begin{array}{llll} \sigma_{s1}: & \xi'_1 = \xi_1, & \eta'_1 = \eta_1 + \xi_1, & \xi'_2 = \xi_2, \quad \eta'_2 = \eta_2 \\ \sigma_{s2}: & \xi'_1 = \xi_1, & \eta'_1 = \eta_1, & \xi'_2 = \xi_2, \quad \eta'_2 = -\xi_2 + \eta_2, \\ \sigma_s: & \xi'_1 = \xi_1, & \eta'_1 = \xi_1 + \eta_1, & \xi'_2 = \xi_2, \quad \eta'_2 = -\xi_2 + \eta_2. \end{array}$$

Note that eigenvalue formulae (2.7) and the Jordan normal form (2.11) tell us that  $\tau_1$  and  $\tau_2$  do not commute, while  $\sigma_{s1}$  and  $\sigma_{s2}$  commute and they are diagonalizable if and only if

$\gamma_s \neq 1/2$ . We further remark that when  $\mu_s$  satisfies (2.10), we have

$$(2.12) \quad \operatorname{Re} \gamma_s \leq 1/2, \operatorname{Im} \gamma_s \geq 0, \quad \text{if } |\mu_s| \geq 1, \operatorname{Im} \mu_s \geq 0;$$

$$(2.13) \quad \operatorname{Re} \gamma_s = 1/2, \operatorname{Im} \gamma_s \geq 0, \quad \gamma_s \neq 1/2, \quad \text{if } |\mu_s| = 1, \operatorname{Im} \mu_s \geq 0, \mu_s \neq 1;$$

$$(2.14) \quad \gamma_s < 1/2, \gamma_s \neq 0, \quad \text{if } \mu_s^2 > 1; \quad \gamma_s = 1/2, \quad \text{if } \mu_s = 1.$$

We have therefore proved the following.

**Proposition 2.2.** *Quadratic surfaces in  $\mathbf{C}^4$  of complex type CR singularity at the origin is classified by (2.6) with  $\gamma_s$  uniquely determined by (2.12)-(2.14).*

The region of eigenvalue  $\mu$ , restricted to  $E := \{|\mu| \geq 1, \operatorname{Im} \mu \geq 1\}$ , can be described as follows: For a Bishop quadric,  $\mu$  is precisely located in  $\omega := \{\mu \in \mathbf{C} : |\mu| = 1\} \cup [1, \infty)$ . The value of  $\mu$  of a quadric of complex type, is precisely located in  $\Omega := E \setminus \{-1\}$ , while  $\omega = \partial E \setminus (-\infty, -1)$ .

In summary, under the condition that no component is a Bishop parabolic quadric or a complex quadric with  $\gamma_s = 1/2$ , we have found linear coordinates for the product quadrics such that the normal forms of  $S$ ,  $T_{ij}$ ,  $\rho$  of the corresponding  $\sigma, \sigma_j, \tau_{ij}, \rho_0$  are given by

$$\begin{aligned} S: \xi'_j &= \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j; \\ T_{ij}: \xi'_j &= \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad \xi'_k = \xi_k, \quad \eta'_k = \eta_k, \quad k \neq j; \\ \rho: \begin{cases} (\xi'_e, \eta'_e, \xi'_h, \eta'_h) &= (\bar{\eta}_e, \bar{\xi}_e, \bar{\xi}_h, \bar{\eta}_h), \\ (\xi'_s, \xi'_{s+s_*}, \eta'_s, \eta'_{s+s_*}) &= (\bar{\xi}_{s+s_*}, \bar{\xi}_s, \bar{\eta}_{s+s_*}, \bar{\eta}_s). \end{cases} \end{aligned}$$

Notice that we can always normalize  $\rho_0$  into the above normal form  $\rho$ .

For various reversible mappings and their relations with general mappings, the reader is referred to [OZ11] for recent results and references therein.

To derive our normal forms, we shall transform  $\{\tau_1, \tau_2, \rho\}$  into a normal form first. We will further normalize  $\{\tau_{1j}, \rho\}$  by using the group of biholomorphic maps that preserve the normal form of  $\{\tau_1, \tau_2, \rho\}$ , i.e. the centralizer of the normal form of  $\{\tau_1, \tau_2, \rho\}$ .

### 3. QUADRICS WITH THE MAXIMUM NUMBER OF DECK TRANSFORMATIONS

In Proposition 2.1, we describe the basic relation between the classification of real manifolds and that of two families of involutions intertwined by an antiholomorphic involution, which is established in [GS15]. As an application, we obtain in this section a normal form for two families of linear involutions and use it to construct the normal form for their associated quadrics. This section also serves an introduction to our approach to find the normal forms of the real submanifolds at least at the formal level. At the end of the section, we will also introduce examples of quadrics of which  $S$  is given by Jordan matrices. The perturbation of such quadrics will not be studied in this paper.

**3.1. Normal form of two families of linear involutions.** To formulate our results, we first discuss the normal forms which we are seeking for the involutions. We are given two families of commuting linear involutions  $\mathcal{T}_1 = \{T_{11}, \dots, T_{1p}\}$  and  $\mathcal{T}_2 = \{T_{21}, \dots, T_{2p}\}$  with  $T_{2j} = \rho T_{1j} \rho$ . Here  $\rho$  is a linear anti-holomorphic involution. We set

$$T_1 = T_{11} \cdots T_{1p}, \quad T_2 = \rho T_1 \rho.$$

We also assume that each  $\text{Fix } T_{1j}$  is a hyperplane and  $\cap \text{Fix } T_{1j}$  has dimension  $p$ . By [GS15, Lemma 2.4], in suitable linear coordinates, each  $T_{1j}$  has the form

$$Z_j: \xi' = \xi, \quad \eta'_i = \eta_i \ (i \neq j), \quad \eta'_j = -\eta_j.$$

Thus by (2.4),

$$(3.1) \quad \dim[\mathfrak{M}_n]_1^{T_i} = p, \quad [\mathfrak{M}_n]_1^{T_i} = [\mathfrak{M}_n]_1^{T_i},$$

$$(3.2) \quad \dim[\mathfrak{M}_n]_1^{T_i} = p, \quad [\mathfrak{M}_n]_1^{T_1} \cap [\mathfrak{M}_n]_1^{T_2} = \{0\}.$$

Recall that  $[\mathfrak{M}_n]_1$  denotes the linear holomorphic functions without constant terms. We would like to find a change of coordinates  $\varphi$  such that  $\varphi^{-1}T_{1j}\varphi$  and  $\varphi^{-1}\rho\varphi$  have a simpler form. We would like to show that two such families of involutions  $\{\mathcal{T}_1, \rho\}$  and  $\{\tilde{\mathcal{T}}_1, \tilde{\rho}\}$  are holomorphically equivalent, if there are normal forms are equivalent under a much smaller set of changes of coordinates, or if they are identical in the ideal situation.

Next, we describe our scheme to derive the normal forms for linear involutions. The scheme to derive the linear normal forms turns out to be essential to derive normal forms for non-linear involutions and the perturbed quadrics. We define

$$S = T_1 T_2.$$

Besides conditions (3.1)-(3.2), we will soon impose condition E that  $S$  has  $2p$  distinct eigenvalues.

We first use a linear map  $\psi$  to diagonalize  $S$  to its normal form

$$\hat{S}: \xi'_j = \mu_j \xi, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p.$$

The choice of  $\psi$  is not unique. We further normalize  $T_1, T_2, \rho$  under linear transformations commuting with  $\hat{S}$ , i.e. the invertible mappings in the *linear centralizer* of  $\hat{S}$ . We use a linear map that commutes with  $\hat{S}$  to transform  $\rho$  into a normal form too, which is still denoted by  $\rho$ . We then use a transformation  $\psi_0$  in the linear centralizer of  $\hat{S}$  and  $\rho$  to normalize the  $T_1, T_2$  into the normal form

$$\hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad 1 \leq j \leq p.$$

Here we require  $\lambda_{2j} = \lambda_{1j}^{-1}$ . Thus  $\mu_j = \lambda_{1j}^2$  for  $1 \leq j \leq p$ , and  $\lambda_{11}, \dots, \lambda_{1p}$  form a complete set of invariants of  $T_1, T_2, \rho$ , provided the normalization satisfies

$$\lambda_{1e} > 1, \quad \text{Im } \lambda_{1h} > 0, \quad \arg \lambda_{1s} \in (0, \pi/2), \quad |\lambda_s| > 1.$$

This normalization will be verified under condition E.

Next we normalize the family  $\mathcal{T}_1$  of linear involutions under mappings in the linear centralizer of  $\hat{T}_1, \rho$ . Let us assume that  $T_1, \rho$  are in the normal forms  $\hat{T}_1, \rho$ . To further normalize the family  $\{\mathcal{T}_1, \rho\}$ , we use the crucial property that  $T_{11}, \dots, T_{1p}$  commute pairwise and each  $T_{1j}$  fixes a hyperplane. This allows us to express the family of involutions via a single linear mapping  $\phi_1$ :

$$T_{1j} = \varphi_1 \phi_1 Z_j \phi_1^{-1} \varphi_1^{-1}.$$

Here the linear mapping  $\varphi_1$  depends only on  $\lambda_1, \dots, \lambda_p$ . Expressing  $\phi_1$  in a non-singular  $p \times p$  constant matrix  $\mathbf{B}$ , the normal form for  $\{T_{11}, \dots, T_{1p}, \rho\}$  consists of invariants  $\lambda_1, \dots, \lambda_p$  and a normal form of  $\mathbf{B}$ . After we obtain the normal form for  $\mathbf{B}$ , we will construct the

normal form of the quadrics by using the realization procedure in the Proposition 2.1 (see the proof in [GS15])

We now carry out the details.

Let  $T_1 = T_{11} \cdots T_{1p}$ ,  $T_2 = \rho T_1 \rho$  and  $S = T_1 T_2$ . Since  $T_i$  and  $\rho$  are involutions, then  $S$  is reversible with respect to  $T_i$  and  $\rho$ , i.e.

$$S^{-1} = T_i^{-1} S T_i, \quad S^{-1} = \rho^{-1} S \rho, \quad T_i^2 = I, \quad \rho^2 = I.$$

Therefore, if  $\kappa$  is an eigenvalue of  $S$  with a (non-zero) eigenvector  $u$ , then

$$S u = \kappa u, \quad S(T_i u) = \kappa^{-1} T_i u, \quad S(\rho u) = \bar{\kappa}^{-1} \rho u, \quad S(\rho T_i u) = \bar{\kappa} \rho T_i u.$$

Following [MW83] and [St07], we will divide eigenvalues of product quadrics that satisfy condition  $E$  into 3 types:  $\mu$  is *elliptic* if  $\mu \neq \pm 1$  and  $\mu$  is real,  $\mu$  is *hyperbolic* if  $|\mu| = 1$  and  $\mu \neq 1$ , and  $\mu$  is *complex* otherwise. The classification of  $\sigma$  into the types corresponds to the classification of the types of complex tangents described in section 2; namely, an elliptic (resp. hyperbolic) complex tangent is tied to a hyperbolic (resp. elliptic) mapping  $\sigma$ .

We first characterize the linear family  $\{T_1, T_2, \rho\}$  that can be realized by a product quadric with  $S$  being diagonal.

**Lemma 3.1.** *Let  $\{T_1, T_2\}$  be a pair of linear involutions on  $\mathbf{C}^{2p}$  satisfying (3.2). Suppose that  $T_2 = \rho T_1 \rho$  for a linear anti-holomorphic involution and  $S = T_1 T_2$  is diagonalizable. Then  $\{T_1, T_2, \rho\}$  is realized by the product of quadrics of type elliptic, hyperbolic, or complex. In particular, if  $S$  has  $2p$  distinct eigenvalues, then 1 and  $-1$  are not eigenvalues of  $S$ .*

*Proof.* The last assertion follows from the first part of the lemma immediately. Thus the following argument does not assume that  $S$  has distinct eigenvalues. Let  $E_i(\nu_i)$  with  $i = 1, \dots, 2p$  be eigenspaces of  $S = T_1 T_2$  with eigenvalues  $\nu_i$ . Thus

$$\mathbf{C}^{2p} = \bigoplus_{i=1}^{2p} E_i(\nu_i), \quad \mathbf{C}^{2p} \ominus E_i(\nu_i) := \bigoplus_{j \neq i} E_j(\nu_j).$$

Fix an  $i$  and denote the corresponding space by  $E(\nu)$ . Since  $\sigma^{-1} = T_1 \sigma T_1$ , then  $T_1 E(\nu) = T_2 E(\nu)$ , which is equal to some invariant space  $E(\nu^{-1})$ . Take an eigenvector  $e \in E(\nu)$  and set  $e' = T_1 e$ .

Let us first show that 1 is not an eigenvalue. Assume for the sake of contradiction that  $E(1)$  is spanned by a (non-zero) eigenvector  $e$ . Then  $T_1$  preserves  $E(1)$ . Otherwise,  $e'$  and  $e$  are independent. Now  $T_2 e = T_1 e = e'$  and  $T_i(e + e') = e' + e$ , which contradicts  $\text{Fix } T_1 \cap \text{Fix } T_2 = \{0\}$ . With  $E(1)$  being preserved by  $T_i$ , we have  $T_i e = \epsilon e$  and  $\epsilon = \pm 1$ , since  $T_i$  are involutions. We have  $\epsilon \neq 1$  since  $\text{Fix } T_1 \cap \text{Fix } T_2 = \{0\}$ . Thus  $T_1 e = -e = T_2 e$ . Then  $\text{Fix } T_1$  and  $\text{Fix } T_2$  are subspaces of  $\mathbf{C}^{2p} \ominus E(1)$  and both are of dimension  $p$ . Hence  $\text{Fix } T_1 \cap \text{Fix } T_2 \neq \{0\}$ , a contradiction.

Since  $S^{-1} = \rho^{-1} S \rho$  and  $S^{-1} = T_i^{-1} S T_i$  then  $T_1$  sends  $E(\nu)$  to some  $E(\nu^{-1})$  as mentioned earlier, while  $\rho$  sends  $E(\nu)$  to some  $E(\bar{\nu}^{-1})$ . In such a way, each of  $T_i, \rho$  yields an involution on the set  $\{E(\nu_1), \dots, E(\nu_{2p})\}$ .

Let  $E_1(-1), \dots, E_k(-1)$  be all spaces invariant by  $T_1$ . Since  $T_2 = T_1 S$ , they are also invariant by  $T_2$ . Then none of the  $k$  spaces is invariant by  $\rho$ . Indeed, if one of them, say  $E_j$  generated by  $e_j$ , is invariant by  $\rho$ , we have  $T_1 e_j = \epsilon e_j$  and  $\rho e_j = b e_j$  with  $\epsilon^2 = 1 = |b|$ . We

get  $T_2 e_j = (\rho T_1 \rho) e_j = \epsilon e_j$  and  $\sigma e_j = e_j$ , which contracts that  $\sigma$  has eigenvalue  $-1$  on  $E_j$ . Furthermore, if  $E(-1)$  is invariant by  $T_1$ , then  $\rho E(-1)$  is also invariant by  $T_1$  as  $T_1 = \rho T_2 \rho$ . Thus we may assume that  $\rho E_j = E_{\ell+j}$  for  $1 \leq j \leq \ell := k/2$ . For each  $j$  with  $1 \leq j \leq \ell$ , either  $T_1 = I = -T_2$  on  $E_j$  and  $T_1 = \rho T_2 \rho = -I$  on  $E_{\ell+j}$ , or  $T_1 = -I$  on  $E_j$  and  $T_1 = I$  on  $E_{\ell+j}$ . Interchanging  $E_j, E_{\ell+j}$  if necessary, we may assume that  $T_1 = I = -T_2$  on  $E_j$  and  $T_1 = -I = -T_2$  on  $E_{\ell+j}$ . We can restrict the involutions  $T_1, T_2, \rho$  on  $\mathbf{C}^2 := E_j \oplus E_{\ell+j}$  as it is invariant by the three involutions. By the realization in [MW83],  $\{T_1, T_2, \rho\}$  is realized by a Bishop quadric; in fact, it is  $Q_\infty$ . Assume now that  $E(-1)$  is not invariant by  $T_1$ . Thus  $T_i$  sends  $E(-1)$  into a different  $\tilde{E}(-1)$ . Assume first that  $E(-1)$  is invariant by  $\rho$ . Then  $\tilde{E}(-1)$  is also invariant by  $\rho$  as  $\rho = T_2 \rho T_1$ . Thus as the previous case  $\{T_1, T_2, \rho\}$ , restricted to  $E(-1) \oplus \tilde{E}(-1)$  is realized by  $Q_\infty$ .

Suppose now that  $\rho$  does not preserve  $E(-1)$ . Recall that we already assume that  $T_1(E(-1)) = \tilde{E}(-1)$  is different from  $E(-1)$ . Let us show that  $\tilde{E}(-1) \neq \rho E(-1)$ . Otherwise, we let  $\tilde{e} = \rho e$  with  $e$  being an eigenvector in  $E(-1)$ . Then  $T_1 e = a \tilde{e}$ . So  $T_2 e = \rho T_1 \rho e = \bar{a}^{-1} \tilde{e}$  and  $T_1 T_2 e = |a|^{-2} e$ . This contracts  $Se = -e$ . We now realize  $E(-1) \oplus \rho E(-1) \oplus \tilde{E}(-1) \oplus \rho \tilde{E}(-1)$  by a product of two copies of  $Q_\infty$  as follows. Take a non-zero vector  $e \in E(-1)$ . Define  $e_1 = e + T_1 e$ . So  $T_1 e_1 = e_1$ ,  $Se_1 = -e_1$ , and  $T_2 e_1 = T_1 S e_1 = -e_1$ . Define  $\tilde{e}_1 = \rho e_1$ ; then  $T_1 \tilde{e}_1 = \rho T_2 \rho \tilde{e}_1 = -\tilde{e}_1$ . Define  $\tilde{e}_2 = e_1 - T_1 e_1$ ; then  $T_1 \tilde{e}_2 = -\tilde{e}_2$  and  $T_2 \tilde{e}_2 = \tilde{e}_2$ . Define  $e_2 = \rho \tilde{e}_2$ ; then  $T_1 e_2 = \rho T_2 \rho e_2 = e_2$ . In coordinates  $z_1 e_1 + w_1 \tilde{e}_1 + z_2 e_2 + w_2 \tilde{e}_2$ , we have  $T_1(z_j) = z_j$  and  $T_1(w_j) = -w_j$  and  $\rho(z_j) = \bar{w}_j$ . Therefore,  $\{T_1, T_2, \rho\}$  is realized by the product of two copies of  $Q_\infty$ .

Consider now the case  $\nu$  is positive and  $\nu \neq 1$ . We have

$$(3.3) \quad T_i: E(\nu) \rightarrow E(\nu^{-1}), \quad i = 1, 2.$$

There are two cases:  $\rho E(\nu) = E(\nu^{-1})$  or  $\rho E(\nu) := \tilde{E}(\nu^{-1}) \neq E(\nu^{-1})$ . For the first case, the family  $\{T_1, T_2, \rho\}$ , restricted to  $E(\nu) \oplus E(\nu^{-1})$ , is realized by an elliptic Bishop quadric  $Q_\gamma$  with  $\gamma \neq 0$ . For the second case, we want to verify that  $\{T_1, T_2, \rho\}$ , restricted to  $E(\nu) \oplus \rho E(\nu) \oplus E(\nu^{-1}) \oplus \rho E(\nu^{-1})$ , is realized by a quadric of complex type singularity. Write  $\nu_1 := \nu = \lambda_1^2$  with  $\lambda_1 > 0$ ,  $\lambda_2 := \lambda_1^{-1}$ , and  $\nu_2 := \nu_1^{-1}$ . Let  $u_1$  be an eigenvector in  $E(\nu)$ . Define  $v_1 = \lambda_1 T_1 u_1 \in E(\nu^{-1})$ . Then  $T_j u_1 = \lambda_j^{-1} v_1$ . Define  $u_2 = \rho u_1$  and  $v_2 = \rho v_1$ . Then  $T_1 u_2 = \rho T_2 \rho u_2 = \rho T_2 u_1 = \lambda_2^{-1} v_2$ . Thus  $\sigma u_j = \nu_j u_j$  and  $\sigma v_j = \nu_j^{-1} v_j$ . We now realize the family of involutions by a quadratic submanifold. For the convenience of the reader, we repeat part of argument in [GS15]; see the paragraph after Proposition 2.1. In coordinates  $\xi_1 u_1 + \xi_2 u_2 + \eta_1 v_1 + \eta_2 v_2$ , we have  $T_i(\xi, \eta) = (\lambda_i \eta, \lambda_i^{-1} \xi)$  and  $\rho(\xi, \eta) = (\bar{\xi}_2, \bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1)$ . Let

$$\begin{aligned} z_j &= \xi_j + \lambda_j \eta_j, & w_j &= \overline{z_j \circ \rho}, & j &= 1, 2; \\ z_3 &= (\eta_1 - \lambda_1^{-1} \xi_1)^2, & z_4 &= (\eta_2 - \lambda_2^{-1} \xi_2)^2. \end{aligned}$$

Expressing  $\xi_j, \eta_j$  via  $(z_1, z_2, w_1, w_2)$ , we obtain

$$z_3 = L_1^2(z_1, z_2, w_1, w_2), \quad z_4 = L_2^2(z_1, z_2, w_1, w_2).$$

Setting  $w_1 = \bar{z}_1$  and  $w_2 = \bar{z}_2$ , we obtain the defining equations of  $M \subset \mathbf{C}^4$  that is a realization of  $\{T_1, T_2, \rho\}$ .

Assume now that  $\nu < 0$  and  $\nu \neq -1$ . We still have (3.3). We want to show that  $\rho(E(\nu)) \neq E(\nu^{-1})$  where  $E(\nu^{-1})$  is in (3.3), i.e. the above second case in  $\nu > 0$  occurs and

the above argument shows that  $\{T_1, T_2, \rho\}$ , restricted to  $E(\nu) \oplus \rho E(\nu) \oplus E(\nu^{-1}) \oplus \rho E(\nu^{-1})$ , is realized by a quadric of complex type singularity. Suppose that  $\rho E(\nu) = E(\nu^{-1})$ . Take  $e \in E(\nu)$ . We can write  $\tilde{e} = \rho e \in E(\nu^{-1})$ . Then  $T_1 e = a\tilde{e}$ . We have  $T_2 e = T_1 S e = \nu a\tilde{e}$  and  $T_2 e = \rho T_1 \rho e = \rho(a^{-1}e) = \bar{a}^{-1}\tilde{e}$ . We obtain  $\nu = |a|^{-2} > 0$ , a contradiction.

Analogously, if  $\nu$  has modulus 1 and is different from  $\pm 1$ , we have two cases:  $\rho E(\nu) = E(\nu^{-1})$  or  $\rho E(\nu) := \tilde{E}(\nu^{-1}) \neq E(\nu^{-1})$ . In the first case,  $\{T_1, T_2, \rho\}$  restricted to the two dimensional subspace is realized by a hyperbolic quadric  $Q_\gamma$  with  $\gamma \neq \infty$ . In the second case its restriction to the 4-dimensional subspace is realized by a quadric of complex CR singularity with  $|\nu| = 1$ . In fact the same argument is valid. Namely, let  $\lambda_1^2 = \nu = \nu_1$ . Let  $\lambda_2 = \lambda_1^{-1}$  and  $\nu_2 = \nu_1^{-1}$ . Take an eigenvector  $e_1 \in E(\nu)$ . Define  $\tilde{e}_1 = \lambda_1 T_1 e_1$ ,  $e_2 = \rho e_1$  and  $\tilde{e}_2 = \rho \tilde{e}_1$ . Then define  $z_j, w_j$  and  $L_j$  as above, which gives us a realization. We leave the details to the reader. Finally, if  $\nu, \bar{\nu}^{-1}, \nu^{-1}, \bar{\nu}$  are distinct, then we have a realization proved in Theorem 3.7 for a general case where all eigenvalues are distinct.  $\square$

Of course, there are non-product quadrics that realize  $\{T_1, T_2, \rho\}$  in Lemma 3.1 and the main purpose of this section is to classify them under condition E. We now assume conditions E and (3.1)-(3.2) for the rest of the section to derive a normal form for  $T_{1j}$  and  $\rho$ .

We need to choose the eigenvectors of  $S$  and their eigenvalues in such a way that  $T_1, T_2$  and  $\rho$  are in a normal form. We will first choose eigenvectors to put  $\rho$  into a normal form. After normalizing  $\rho$ , we will then choose eigenvectors to normalize  $T_1$  and  $T_2$ .

First, let us consider an elliptic eigenvalue  $\mu_e$ . Let  $u$  be an eigenvector of  $\mu_e$ . Then  $u$  and  $v = \rho(u)$  satisfy

$$(3.4) \quad S(v) = \mu_e^{-1}v, \quad T_j(u) = \lambda_j^{-1}v, \quad \mu_e = \lambda_1 \lambda_2^{-1}.$$

Now  $T_2(u) = \rho T_1 \rho(u)$  implies that

$$\lambda_2 = \bar{\lambda}_1^{-1}, \quad \mu_e = |\lambda_1|^2.$$

Replacing  $(u, v)$  by  $(cu, \bar{c}v)$ , we may assume that  $\lambda_1 > 0$  and  $\lambda_2 = \lambda_1^{-1}$ . Replacing  $(u, v)$  by  $(v, u)$  if necessary, we may further achieve

$$\rho(u) = v, \quad \lambda_1 = \lambda_e > 1, \quad \mu_e = \lambda_e^2 > 1.$$

We still have the freedom to replace  $(u, v)$  by  $(ru, rv)$  for  $r \in \mathbf{R}^*$ , while preserving the above conditions.

Next, let  $\mu_h$  be a hyperbolic eigenvalue of  $S$  and  $S(u) = \mu_h u$ . Then  $u$  and  $v = T_1(u)$  satisfy

$$\rho(u) = au, \quad \rho(v) = bv, \quad |a| = |b| = 1.$$

Replacing  $(u, v)$  by  $(cu, v)$ , we may assume that  $a = 1$ . Now  $T_2(v) = \rho T_1 \rho(v) = \bar{b}u$ . To obtain  $b = 1$ , we replace  $(u, v)$  by  $(u, b^{-1/2}v)$ . This give us (3.4) with  $|\lambda_j| = 1$ . Replacing  $(u, v)$  by  $(v, u)$  if necessary, we may further achieve

$$\rho(u) = u, \quad \rho(v) = v, \quad \lambda_1 = \lambda_h, \quad \mu_h = \lambda_h^2, \quad \arg \lambda_h \in (0, \pi/2).$$

Again, we have the freedom to replace  $(u, v)$  by  $(ru, rv)$  for  $r \in \mathbf{R}^*$ , while preserving the above conditions.

Finally, we consider a complex eigenvalue  $\mu_s$ . Let  $S(u) = \mu_s u$ . Then  $\tilde{u} = \rho(u)$  satisfies  $S(\tilde{u}) = \bar{\mu}_s^{-1} \tilde{u}$ . Let  $u^* = T_1(u)$  and  $\tilde{u}^* = \rho(u^*)$ . Then  $S(u^*) = \mu_s^{-1} u^*$  and  $S(\tilde{u}^*) = \bar{\mu}_s \tilde{u}^*$ . We change eigenvectors by

$$(u, \tilde{u}, u^*, \tilde{u}^*) \rightarrow (u, \tilde{u}, cu^*, \bar{c}\tilde{u}^*)$$

so that

$$\begin{aligned} \rho(u) &= \tilde{u}, \quad \rho(u^*) = \tilde{u}^*, \\ T_j(u) &= \lambda_j^{-1} u^*, \quad T_j(\tilde{u}) = \bar{\lambda}_j \tilde{u}^*, \quad \lambda_2 = \lambda_1^{-1}. \end{aligned}$$

Note that  $S(u) = \lambda_1^2 u$ ,  $S(u^*) = \lambda_1^{-2} u^*$ ,  $S(\tilde{u}) = \bar{\lambda}_1^{-2} \tilde{u}$ , and  $S(\tilde{u}^*) = \bar{\lambda}_1^2 \tilde{u}^*$ . Replacing  $(u, \tilde{u}, u^*, \tilde{u}^*)$  by  $(u^*, \tilde{u}^*, u, \tilde{u})$  changes the argument and the modulus of  $\lambda_1$  as  $\lambda_1^{-1}$  becomes  $\lambda_1$ . Replacing them by  $(\tilde{u}, u, \tilde{u}^*, u^*)$  changes only the modulus as  $\lambda_1$  becomes  $\bar{\lambda}_1^{-1}$  and then replacing them by  $(u^*, \tilde{u}^*, -u, -\tilde{u})$  changes the sign of  $\lambda_1$ . Therefore, we may achieve

$$\mu_s = \lambda_s^2, \quad \lambda_1 = \lambda_s, \quad \arg \gamma_s \in (0, \pi/2), \quad |\lambda_s| > 1.$$

We still have the freedom to replace  $(u, u^*, \tilde{u}, \tilde{u}^*)$  by  $(cu, cu^*, \bar{c}\tilde{u}, \bar{c}\tilde{u}^*)$ .

We summarize the above choice of eigenvectors and their corresponding coordinates. First,  $S$  has distinct eigenvalues

$$\lambda_e^2 = \bar{\lambda}_e^2, \quad \lambda_e^{-2}; \quad \lambda_h^2, \quad \bar{\lambda}_h^2 = \lambda_h^{-2}; \quad \lambda_s^2, \quad \lambda_s^{-2}, \quad \bar{\lambda}_s^{-2}, \quad \bar{\lambda}_s^2.$$

Also,  $S$  has linearly independent eigenvectors satisfying

$$\begin{aligned} Su_e &= \lambda_e^2 u_e, \quad Su_e^* = \lambda_e^{-2} u_e^*, \\ Sv_h &= \lambda_h^2 v_h, \quad Sv_h^* = \lambda_h^{-2} v_h^*, \\ Sw_s &= \lambda_s^2 w_s, \quad Sw_s^* = \lambda_s^{-2} w_s^*, \quad S\tilde{w}_s = \bar{\lambda}_s^{-2} \tilde{w}_s, \quad S\tilde{w}_s^* = \bar{\lambda}_s^2 \tilde{w}_s^*. \end{aligned}$$

Furthermore, the  $\rho$ ,  $T_1$ , and the chosen eigenvectors of  $S$  satisfy

$$\begin{aligned} \rho u_e &= u_e^*, \quad T_1 u_e = \lambda_e^{-1} u_e^*, \\ \rho v_h &= v_h, \quad \rho v_h^* = v_h^*, \quad T_1 v_h = \lambda_h^{-1} v_h^*, \\ \rho w_s &= \tilde{w}_s, \quad \rho w_s^* = \tilde{w}_s^*, \quad T_1 w_s = \lambda_s^{-1} w_s^*, \quad T_1 \tilde{w}_s = \bar{\lambda}_s \tilde{w}_s^*. \end{aligned}$$

For normalization, we collect elliptic eigenvalues  $\mu_e$  and  $\mu_e^{-1}$ , hyperbolic eigenvalues  $\mu_h$  and  $\mu_h^{-1}$ , and complex eigenvalues in  $\mu_s, \mu_s^{-1}, \bar{\mu}_s^{-1}$  and  $\bar{\mu}_s$ . We put them in the order

$$\begin{aligned} \mu_e &= \bar{\mu}_e, \quad \mu_{p+e} = \mu_e^{-1}, \\ \mu_h, \quad \mu_{p+h_*+h} &= \bar{\mu}_h, \\ \mu_s, \quad \mu_{s+s_*} &= \bar{\mu}_s^{-1}, \quad \mu_{p+s} = \mu_s^{-1}, \quad \mu_{p+s_*+s} = \bar{\mu}_s. \end{aligned}$$

Here and throughout the paper the ranges of subscripts  $e, h, s$  are restricted to

$$1 \leq e \leq e_*, \quad e_* < h \leq e_* + h_*, \quad e_* + h_* < s \leq p - s_*.$$

Thus  $e_* + h_* + 2s_* = p$ . Using the new coordinates

$$\sum (\xi_e u_e + \eta_e u_e^*) + \sum (\xi_h v_h + \eta_h v_h^*) + \sum (\xi_s w_s + \xi_{s+s_*} \tilde{w}_s + \eta_s w_s^* + \eta_{s+s_*} \tilde{w}_s^*),$$

we have normalized  $\sigma, T_1, T_2$  and  $\rho$ . In summary, we have the following normal form.



**Lemma 3.2.** *Let  $T_1, T_2$  be linear holomorphic involutions on  $\mathbf{C}^n$  that satisfy (3.2). Then  $n = 2p$  and  $\dim[\mathfrak{M}_n]_1^{T_i} = p$ . Suppose that  $T_2 = \rho_0 T_1 \rho_0$  for some anti-holomorphic linear involution  $\rho_0$ . Assume that  $S = T_1 T_2$  has  $n$  distinct eigenvalues. There exists a linear change of holomorphic coordinates that transforms  $T_1, T_2, S, \rho_0$  simultaneously into the normal forms  $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$ :*

$$(3.5) \quad \hat{T}_1: \xi'_j = \lambda_j \eta_j, \quad \eta'_j = \lambda_j^{-1} \xi, \quad 1 \leq j \leq p;$$

$$(3.6) \quad \hat{T}_2: \xi'_j = \lambda_j^{-1} \eta_j, \quad \eta'_j = \lambda_j \xi_j, \quad 1 \leq j \leq p;$$

$$(3.7) \quad \hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p;$$

$$(3.8) \quad \rho: \begin{cases} \xi'_e = \bar{\eta}_e, & \eta'_e = \bar{\xi}_e, \\ \xi'_h = \bar{\xi}_h, & \eta'_h = \bar{\eta}_h, \\ \xi'_s = \bar{\xi}_{s+s_*}, & \xi'_{s+s_*} = \bar{\xi}_s, \\ \eta'_s = \bar{\eta}_{s+s_*}, & \eta'_{s+s_*} = \bar{\eta}_s. \end{cases}$$

Moreover, the eigenvalues  $\mu_1, \dots, \mu_p$  satisfy

$$(3.9) \quad \mu_j = \lambda_j^2, \quad 1 \leq j \leq p;$$

$$(3.10) \quad \lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \bar{\lambda}_s^{-1};$$

$$(3.11) \quad \arg \lambda_h \in (0, \pi/2), \quad \arg \lambda_s \in (0, \pi/2);$$

$$(3.12) \quad \lambda_{e'} < \lambda_{e'+1}, \quad 0 < \arg \lambda_{h'} < \arg \lambda_{h'+1} < \pi/2;$$

$$(3.13) \quad \arg \lambda_{s'} < \arg \lambda_{s'+1}, \text{ or } \arg \lambda_{s'} = \arg \lambda_{s'+1} \text{ and } |\lambda_{s'}| < |\lambda_{s'+1}|.$$

Here  $1 \leq e' < e_*$ ,  $e_* < h' < e_* + h_*$ , and  $e_* + h_* < s' < p - s_*$ . And  $1 \leq e \leq e_*$ ,  $e_* < h \leq e_* + h_*$ , and  $e_* + h_* < s \leq p - s_*$ . If  $\tilde{S}$  is also in the normal form (3.7) for possible different eigenvalues  $\tilde{\mu}_1, \dots, \tilde{\mu}_p$  satisfying (3.9)-(3.13), then  $S$  and  $\tilde{S}$  are equivalent if and only if their eigenvalues are identical.

The above normal form of  $\rho$  will be fixed for the rest of paper. Note that in case of non-linear involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  of which the linear part are given by  $\{T_{11}, \dots, T_{1p}, \rho\}$  we can always linearize  $\rho$  first under a holomorphic map of which the linear part at the origin is described in above normalization for the linear part of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ . Indeed, we may assume that the linear part of the latter family is already in the normal form. Then  $\psi = \frac{1}{2}(I + (L\rho) \circ \rho)$  is tangent to the identity and  $(L\rho) \circ \psi \circ \rho = \psi$ , i.e.  $\psi$  transforms  $\rho$  into  $\bar{L}\rho$  while preserving the linear parts of  $\tau_{11}, \dots, \tau_{1p}$ . Therefore in the non-linear case, we can assume that  $\rho$  is given by the above normal form. The above lemma tells us the ranges of eigenvalues  $\mu_e, \mu_h$  and  $\mu_s$  that can be realized by quadrics that satisfy conditions E and (3.1)-(3.2).

Having normalized  $T_1$  and  $\rho$ , we want to further normalize  $\{T_{11}, \dots, T_{1p}\}$  under linear maps that preserve the normal forms of  $\hat{T}_1$  and  $\rho$ . We know that the composition of  $T_{1j}$  is in the normal form, i.e.

$$(3.14) \quad T_{11} \cdots T_{1p} = \hat{T}_1$$

is given in Lemma 3.2. We first find an expression for all  $T_{1j}$  that commute pairwise and satisfy (3.14), by using invariant and skew-invariant functions of  $\hat{T}_1$ . Let

$$(\xi, \eta) = \varphi_1(z^+, z^-)$$

be defined by

$$(3.15) \quad z_e^+ = \xi_e + \lambda_e \eta_e, \quad z_e^- = \eta_e - \lambda_e^{-1} \xi_e,$$

$$(3.16) \quad z_h^+ = \xi_h + \lambda_h \eta_h, \quad z_h^- = \eta_h - \bar{\lambda}_h \xi_h,$$

$$(3.17) \quad z_s^+ = \xi_s + \lambda_s \eta_s, \quad z_s^- = \eta_s - \lambda_s^{-1} \xi_s,$$

$$(3.18) \quad z_{s+s_*}^+ = \xi_{s+s_*} + \bar{\lambda}_s^{-1} \eta_{s+s_*}, \quad z_{s+s_*}^- = \eta_{s+s_*} - \bar{\lambda}_s \xi_{s+s_*}.$$

In  $(z^+, z^-)$  coordinates,  $\varphi_1^{-1} \hat{T}_1 \varphi_1$  becomes

$$Z: z^+ \rightarrow z^+, \quad z^- \rightarrow -z^-.$$

We decompose  $Z = Z_1 \cdots Z_p$  by using

$$Z_j: (z^+, z^-) \rightarrow (z^+, z_1^-, \dots, z_{j-1}^-, -z_j^-, z_{j+1}^-, \dots, z_p^-).$$

To keep simple notation, let us use the same notions  $x, y$  for a linear transformation  $y = A(x)$  and its matrix representation:

$$A: x \rightarrow \mathbf{A}x.$$

The following lemma, which can be verified immediately, shows the advantages of coordinates  $z^+, z^-$ .

**Lemma 3.3.** *The linear centralizer of  $Z$  is the set of mappings of the form*

$$(3.19) \quad \phi: (z^+, z^-) \rightarrow (\mathbf{A}z^+, \mathbf{B}z^-),$$

where  $\mathbf{A}, \mathbf{B}$  are constant and possibly singular matrices. Let  $\nu$  be a permutation of  $\{1, \dots, p\}$ . Then  $Z_j \phi = \phi Z_{\nu(j)}$  for all  $j$  if and only if  $\phi$  has the above form with  $\mathbf{B} = \text{diag}_\nu \mathbf{d}$ . Here

$$(3.20) \quad \text{diag}_\nu(d_1, \dots, d_p) := (b_{ij})_{p \times p}, \quad b_{j\nu(j)} = d_j, \quad b_{jk} = 0 \text{ if } k \neq \nu(j).$$

In particular, the linear centralizer of  $\{Z_1, \dots, Z_p\}$  is the set of mappings (3.19) in which  $\mathbf{B}$  are diagonal.

To continue our normalization for the family  $\{T_{1j}\}$ , we note that  $\varphi_1^{-1} T_{11} \varphi_1, \dots, \varphi_1^{-1} T_{1p} \varphi_1$  generate an abelian group of  $2^p$  involutions and each of these  $p$  generators fixes a hyperplane. By [GS15, Lemma 2.4], there is a linear transformation  $\phi_1$  such that

$$\varphi_1^{-1} T_{1j} \varphi_1 = \phi_1 Z_j \phi_1^{-1}, \quad 1 \leq j \leq p.$$

Computing two compositions on both sides, we see that  $\phi_1$  must be in the linear centralizer of  $Z$ . Thus, it is in the form (3.19). Of course,  $\phi_1$  is not unique;  $\tilde{\phi}_1$  is another such linear map for the same  $T_{1j}$  if and only if  $\tilde{\phi}_1 = \phi_1 \psi_1$  with  $\psi_1 \in \mathcal{C}(Z_1, \dots, Z_p)$ . By (3.19), we may restrict ourselves to  $\phi_1$  given by

$$(3.21) \quad \phi_1: (z^+, z^-) \rightarrow (z^+, \mathbf{B}z^-).$$

Then  $\tilde{\phi}_1$  yields the same family  $\{T_{1j}\}$  if and only if its corresponding matrix  $\tilde{\mathbf{B}} = \mathbf{B}\mathbf{D}$  for a diagonal matrix  $\mathbf{D}$ .

In the above we have expressed all  $T_{11}, \dots, T_{1p}$  via equivalence classes of matrices. It will be convenient to restate them via matrices.

For simplicity,  $T_i$  and  $S$  denote  $\hat{T}_i, \hat{S}$ , respectively. In matrices, we write

$$T_1: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \mathbf{T}_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \rho: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \boldsymbol{\rho} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix}, \quad S: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \mathbf{S} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Recall that the bold faced  $\mathbf{A}$  represents a linear map  $A$ . Then

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_1^{-1} & \mathbf{0} \end{pmatrix}_{2p \times 2p}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}_{2p \times 2p}.$$

We will abbreviate

$$\boldsymbol{\xi}_{e_*} = (\xi_1, \dots, \xi_{e_*}), \quad \boldsymbol{\xi}_{h_*} = (\xi_{e_*+1}, \dots, \xi_{e_*+h_*}), \quad \boldsymbol{\xi}_{2s_*} = (\xi_{e_*+h_*+1}, \dots, \xi_p).$$

We use the same abbreviation for  $\eta$ . Then  $(\boldsymbol{\xi}_{e_*}, \boldsymbol{\eta}_{e_*})$ ,  $(\boldsymbol{\xi}_{h_*}, \boldsymbol{\eta}_{h_*})$ , and  $(\boldsymbol{\xi}_{2s_*}, \boldsymbol{\eta}_{2s_*})$  subspaces are invariant under  $T_{1j}$ ,  $T_1$ , and  $\rho$ . We also denote by  $T_1^{e_*}, T_1^{h_*}, T_1^{s_*}$  the restrictions of  $T_1$  to these subspaces. Define analogously for the restrictions of  $\rho, S$  to these subspaces. Define diagonal matrices  $\mathbf{\Lambda}_{1e_*}, \mathbf{\Lambda}_{1h_*}, \mathbf{\Lambda}_{1s_*}$ , of size  $e_* \times e_*$ ,  $h_* \times h_*$  and  $s_* \times s_*$  respectively, by

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \mathbf{\Lambda}_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{1h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}}^{-1} \end{pmatrix}, \quad \overline{\mathbf{\Lambda}_1} = \begin{pmatrix} \mathbf{\Lambda}_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{1h_*}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*}^{-1} \end{pmatrix}.$$

Thus, we can express  $T_1^{s_*}$  and  $S^{s_*}$  in  $(2s_*) \times (2s_*)$  matrices

$$\mathbf{T}_1^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}}^{-1} \\ \mathbf{\Lambda}_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S}^{s_*} = \begin{pmatrix} \mathbf{\Lambda}_{1s_*}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}}^{-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*}^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}}^2 \end{pmatrix}.$$

Let  $\mathbf{I}_k$  denote the  $k \times k$  identity matrix. With the abbreviation, we can express  $\rho$  as

$$\boldsymbol{\rho}^{e_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{e_*} \\ \mathbf{I}_{e_*} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\rho}^{h_*} = \mathbf{I}_{2h_*},$$

$$\boldsymbol{\rho}^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} \end{pmatrix}.$$

Note that  $\rho$  is anti-holomorphic linear transformation. If  $A$  is a complex linear transformation, in  $(\xi, \eta)$  coordinates the matrix of  $\rho A$  is  $\boldsymbol{\rho} \overline{\mathbf{A}}$ , i.e.

$$\rho A: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \boldsymbol{\rho} \overline{\mathbf{A}} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix}$$

with

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{I}_{e*} & 0 & 0 & 0 \\ 0 & \mathbf{I}_{h*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{s*} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{s*} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{I}_{e*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{h*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{s*} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{s*} & 0 \end{pmatrix}.$$

For an invertible  $p \times p$  matrix  $\mathbf{A}$ , let us define an  $n \times n$  matrix  $\mathbf{E}_{\mathbf{A}}$  by

$$(3.22) \quad \mathbf{E}_{\mathbf{A}} := \frac{1}{2} \begin{pmatrix} \mathbf{I}_p & -\mathbf{A} \\ \mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}, \quad \mathbf{E}_{\mathbf{A}}^{-1} = \begin{pmatrix} \mathbf{I}_p & \mathbf{A} \\ -\mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}.$$

For a  $p \times p$  matrix  $\mathbf{B}$ , we define

$$\mathbf{B}_* := \begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{B} \end{pmatrix}.$$

Therefore, we can express

$$(3.23) \quad \mathbf{T}_{1j} = \mathbf{E}_{\mathbf{A}_1} \mathbf{B}_* \mathbf{Z}_j \mathbf{B}_*^{-1} \mathbf{E}_{\mathbf{A}_1}^{-1}, \quad \mathbf{T}_{2j} = \rho \overline{\mathbf{T}_{1j}} \rho,$$

$$(3.24) \quad \mathbf{Z}_j = \text{diag}(1, \dots, 1, -1, 1, \dots, 1).$$

Here  $-1$  is at the  $(p+j)$ -th place. By Lemma 3.3,  $\mathbf{B}$  is uniquely determined up to equivalence relation via diagonal matrices  $\mathbf{D}$ :

$$(3.25) \quad \mathbf{B} \sim \mathbf{B}\mathbf{D}.$$

We have expressed all  $\{T_{11}, \dots, T_{1p}, \rho\}$  for which  $\hat{T}_1 = T_{11} \cdots T_{1p}$  and  $\rho$  are in the normal forms in Lemma 3.2 and we have found an equivalence relation to classify the involutions. Let us summarize the results in a lemma.

**Lemma 3.4.** *Let  $\{T_{11}, \dots, T_{1p}, \rho\}$  be the involutions of a quadric manifold  $M$ . Assume that  $S = T_1 \rho T_1 \rho$  has distinct eigenvalues. Then in suitable linear  $(\xi, \eta)$  coordinates,  $T_{11}, \dots, T_{1p}$  are given by (3.23), while  $T_{11} \cdots T_{1p} = \hat{T}_1$  and  $\rho$  are given by (3.5) and (3.8), respectively. Moreover,  $\mathbf{B}$  in (3.23) is uniquely determined by the equivalence relation (3.25) for diagonal matrices  $\mathbf{D}$ .*

We remind the reader that we divide the classification for  $\{T_{11}, \dots, T_{1p}, \rho\}$  into two steps. We have obtained the classification for the composition  $T_{11} \cdots T_{1p} = \hat{T}_1$  and  $\rho$  in Lemma 3.2. Having found all  $\{T_{11}, \dots, T_{1p}, \rho\}$  and an equivalence relation, we are ready to reduce their classification to an equivalence problem that involves two dilatations and a coordinate permutation.

**Lemma 3.5.** *Let  $\{T_{i1}, \dots, T_{ip}, \rho\}$  be given by (3.23). Suppose that  $\hat{T}_1 = T_{11} \cdots T_{1p}$ ,  $\rho$ ,  $\hat{T}_2 = \rho \hat{T}_1 \rho$ , and  $\hat{S} = \hat{T}_1 \hat{T}_2$  have the forms in Lemma 3.2. Suppose that  $\hat{S}$  has distinct eigenvalues. Let  $\{\hat{T}_{11}, \dots, \hat{T}_{1p}, \rho\}$  be given by (3.23) where  $\lambda_j$  are unchanged and  $\mathbf{B}$  is*

replaced by  $\hat{\mathbf{B}}$ . Suppose that  $R^{-1}T_{1j}R = \hat{T}_{1\nu(j)}$  for all  $j$  and  $R\rho = \rho R$ . Then the matrix of  $R$  is  $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$  with  $\mathbf{a} = (\mathbf{a}_{e_*}, \mathbf{a}_{h_*}, \mathbf{a}_{s_*}, \mathbf{a}'_{s_*})$ , while  $\mathbf{a}$  satisfies the reality condition

$$(3.26) \quad \mathbf{a}_{e_*} \in (\mathbf{R}^*)^{e_*}, \quad \mathbf{a}_{h_*} \in (\mathbf{R}^*)^{h_*}, \quad \overline{\mathbf{a}_{s_*}} = \mathbf{a}'_{s_*} \in (\mathbf{C}^*)^{s_*}.$$

Moreover, there exists  $\mathbf{d} \in (\mathbf{C}^*)^p$  such that

$$(3.27) \quad \hat{\mathbf{B}} = (\text{diag } \mathbf{a})^{-1} \mathbf{B} (\text{diag}_\nu \mathbf{d}), \quad \text{i.e.,} \quad a_i^{-1} b_{i\nu^{-1}(j)} d_{\nu^{-1}(j)} = \hat{b}_{ij}, \quad 1 \leq i, j \leq p.$$

Conversely, if  $\mathbf{a}, \mathbf{d}$  satisfy (3.26) and (3.27), then  $R^{-1}T_{1j}R = \hat{T}_{1\nu(j)}$  and  $R\rho = \rho R$ .

*Proof.* Suppose that  $R^{-1}T_{1j}R = \hat{T}_{1\nu(j)}$  and  $R\rho = \rho R$ . Then  $R^{-1}\hat{T}_1R = \hat{T}_1$  and  $R^{-1}\hat{S}R = \hat{S}$ . The latter implies that the matrix of  $R$  is diagonal. The former implies that

$$R: \xi'_j = a_j \xi_j, \quad \eta'_j = a_j \eta_j$$

with  $a_j \in \mathbf{C}^*$ . Now  $R\rho = \rho R$  implies (3.26). We express  $R^{-1}T_{1j}R = \hat{T}_{1\nu(j)}$  via matrices:

$$(3.28) \quad \mathbf{E}_{\Lambda_1} \hat{\mathbf{B}}_* \mathbf{Z}_{\nu(j)} \hat{\mathbf{B}}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} = \mathbf{R}^{-1} \mathbf{E}_{\Lambda_1} \mathbf{B}_* \mathbf{Z}_j \mathbf{B}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} \mathbf{R}.$$

In view of formula (3.22), we see that  $\mathbf{E}_{\Lambda_1}$  commutes with  $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$ . The above is equivalent to that  $\psi := \mathbf{B}_*^{-1} \mathbf{R} \hat{\mathbf{B}}_*$  satisfies  $\mathbf{Z}_{\nu(j)} = \psi^{-1} \mathbf{Z}_j \psi$ . By Lemma 3.3 we obtain  $\psi = \text{diag}(\mathbf{A}, \text{diag}_\nu \mathbf{d})$ . This shows that

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \text{diag}_\nu \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} \text{diag } \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \text{diag } \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{B}} \end{pmatrix}.$$

The matrices on diagonal yield  $\mathbf{A} = \text{diag } \mathbf{a}$  and (3.27). The lemma is proved.  $\square$

Lemma 3.5 does not give us an explicit description of the normal form for the families of involutions  $\{T_{11}, \dots, T_{1p}, \rho\}$ . Nevertheless by the lemma, we can always choose a  $\nu$  and  $\text{diag } \mathbf{d}$  such that the diagonal elements of  $\hat{\mathbf{B}}$ , corresponding to  $\{\hat{T}_{1\nu(1)}, \dots, \hat{T}_{1\nu(p)}, \rho\}$ , are 1.

**Remark 3.6.** In what follows, we will fix a  $\mathbf{B}$  and its associated  $\{\mathcal{T}_1, \rho\}$  to further study our normal form problems.

**3.2. Normal form of the quadrics.** We now use the matrices  $\mathbf{B}$  to express the normal form for the quadratic submanifolds. Here we follow the realization procedure in Proposition 2.1. We will use the coordinates  $z^+, z^-$  again to express invariant functions of  $T_{1j}$  and use them to construct the corresponding quadric. We will then pull back the quadric to the  $(\xi, \eta)$  coordinates and then to the  $z, \bar{z}$  coordinates to achieve the final normal form of the quadrics.

We return to the construction of invariant and skew-invariant functions  $z^+, z^-$  in (3.15)-(3.18) when  $\mathbf{B}$  is the identity matrix. For a general  $\mathbf{B}$ , we define  $\Phi_1$  and the matrix  $\Phi_1^{-1}$  by

$$\Phi_1(Z^+, Z^-) = (\xi, \eta), \quad \Phi_1^{-1} := \mathbf{B}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} = \begin{pmatrix} \mathbf{I} & \Lambda_1 \\ -\mathbf{B}^{-1} \Lambda_1^{-1} & \mathbf{B}^{-1} \end{pmatrix}.$$

Note that  $Z^+ = z^+$  and  $\Phi_1^{-1} T_{1j} \Phi_1 = Z_j$ . The  $Z^+, Z_i^-$  with  $i \neq j$  are invariant functions of  $T_{1j}$ , while  $Z_j^-$  is a skew-invariant function of  $T_{1j}$ . They can be written as

$$(3.29) \quad Z^+ = \xi + \Lambda_1 \eta, \quad Z^- = \mathbf{B}^{-1} (-\Lambda_1^{-1} \xi + \eta).$$

Therefore, the invariant functions of  $\mathcal{T}_1$  are generated by

$$Z_j^+ = \xi_j + \lambda_j \eta_j, \quad (Z_j^-)^2 = (\tilde{\mathbf{B}}_j(-\mathbf{\Lambda}_1^{-1}\xi + \eta))^2, \quad 1 \leq j \leq p.$$

Here  $\tilde{\mathbf{B}}_j$  is the  $j$ th row of  $\mathbf{B}^{-1}$ . The invariant (holomorphic) functions of  $\mathcal{T}_2$  are generated by

$$(3.30) \quad W_j^+ = \overline{Z_j^+ \circ \rho}, \quad (W_j^-)^2 = (\overline{Z_j^- \circ \rho})^2, \quad 1 \leq j \leq p.$$

Here  $W_j^- = \overline{Z_j^- \circ \rho}$ . We will soon verify that

$$m: (\xi, \eta) \rightarrow (z', w') = (Z^+(\xi, \eta), W^+(\xi, \eta))$$

is biholomorphic. A straightforward computation shows that  $m\rho m^{-1}$  equals

$$\rho_0: (z', w') \rightarrow (\overline{w'}, \overline{z'}).$$

We define

$$M: z''_{p+j} = (Z_j^- \circ m^{-1}(z', \overline{z'}))^2.$$

We want to find a simpler expression for  $M$ . We first separate  $B$  from  $Z^-$  by writing

$$(3.31) \quad \hat{\mathbf{Z}}^- := (-\mathbf{\Lambda}_1^{-1} \mathbf{I}), \quad \mathbf{Z}^- = \mathbf{B}^{-1} \hat{\mathbf{Z}}^-.$$

Note that  $m$  does not depend on  $\mathbf{B}$ . To compute  $\hat{\mathbf{Z}}^- \circ m^{-1}$ , we will use matrix expressions for  $(\xi_{e_*}, \eta_{e_*})$ ,  $(\xi_{h_*}, \eta_{h_*})$  and  $(\xi_{2s_*}, \eta_{2s_*})$  subspaces. Let  $m_{e_*}, m_{h_*}, m_{s_*}$  be the restrictions  $m$  to these subspaces. In the matrix form, we have by (3.30)

$$\mathbf{W}^+ = \overline{\mathbf{Z}^+ \rho}, \quad \mathbf{W}^- = \overline{\mathbf{Z}^- \rho}.$$

Recall that  $\mathbf{\Lambda}_1 = \text{diag}(\mathbf{\Lambda}_{e_*}, \mathbf{\Lambda}_{h_*}, \mathbf{\Lambda}_{1s_*}, \overline{\mathbf{\Lambda}_{1s_*}}^{-1})$ . Thus

$$\begin{aligned} \mathbf{m}_{e_*} &= \begin{bmatrix} \mathbf{I} & \mathbf{\Lambda}_{1e_*} \\ \mathbf{\Lambda}_{1e_*} & \mathbf{I} \end{bmatrix}, \quad \mathbf{m}_{e_*}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{\Lambda}_{1e_*} \\ -\mathbf{\Lambda}_{1e_*} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \mathbf{\Lambda}_{1e_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{\Lambda}_{1e_*}^2)^{-1} \end{bmatrix}, \\ \mathbf{m}_{h_*} &= \begin{bmatrix} \mathbf{I} & \mathbf{\Lambda}_{1h_*} \\ \mathbf{I} & \mathbf{\Lambda}_{1h_*}^{-1} \end{bmatrix}, \quad \mathbf{m}_{h_*}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{\Lambda}_{1h_*}^2 \\ -\mathbf{\Lambda}_{1h_*} & \mathbf{\Lambda}_{1h_*} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \mathbf{\Lambda}_{1h_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{\Lambda}_{1h_*}^2)^{-1} \end{bmatrix}, \\ \mathbf{m}_{s_*} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}} \\ \mathbf{I} & \mathbf{0} & \mathbf{\Lambda}_{1s_*}^{-1} & \mathbf{0} \end{bmatrix}, \\ \mathbf{m}_{s_*}^{-1} &= \begin{bmatrix} \mathbf{\Lambda}_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & -\mathbf{\Lambda}_{1s_*} \\ \mathbf{0} & \overline{\mathbf{\Lambda}_{1s_*}} & -\overline{\mathbf{\Lambda}_{1s_*}}^{-1} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s_*} & \mathbf{0} \\ \mathbf{0} & -\overline{\mathbf{L}_{s_*}} \end{bmatrix}, \\ \mathbf{L}_{s_*} &= \begin{bmatrix} (\mathbf{\Lambda}_{1s_*}^{-1} - \mathbf{\Lambda}_{1s_*})^{-1} & \mathbf{0} \\ \mathbf{0} & (\overline{\mathbf{\Lambda}_{1s_*}} - \overline{\mathbf{\Lambda}_{1s_*}}^{-1})^{-1} \end{bmatrix}. \end{aligned}$$

Note that  $\mathbf{I} - \Lambda_1^2$  is diagonal. Using (3.31) and the above formulae, the matrices of  $\hat{Z}_{e*}^{-1} \circ m^{-1}$ ,  $\hat{Z}_{h*}^{-1} \circ m^{-1}$ , and  $\hat{Z}_{s*}^{-1} \circ m^{-1}$  are respectively given by

$$\begin{aligned}\hat{Z}_{e*}^{-1} \mathbf{m}_{e*}^{-1} &= \mathbf{L}_{e*} [\mathbf{I} \quad -2(\Lambda_{1e*} + \Lambda_{1e*}^{-1})^{-1}], \\ \mathbf{L}_{e*} &= (\mathbf{I} - \Lambda_{1e*}^2)^{-1} (-\Lambda_{1e*} - \Lambda_{1e*}^{-1}), \\ \hat{Z}_{h*}^{-1} \mathbf{m}_{h*}^{-1} &= \mathbf{L}_{h*} [\mathbf{I} \quad -2\Lambda_{1h*}(\Lambda_{1h*} + \Lambda_{1h*}^{-1})^{-1}], \\ \mathbf{L}_{h*} &= (\mathbf{I} - \Lambda_{1h*}^2)^{-1} (-\Lambda_{1h*} - \Lambda_{1h*}^{-1}), \\ \hat{Z}_{s*}^{-1} \mathbf{m}_{s*}^{-1} &= \begin{bmatrix} -\mathbf{I} - \Lambda_{1s*}^{-2} & \mathbf{0} & \mathbf{0} & 2\mathbf{I} \\ \mathbf{0} & -\mathbf{I} - \bar{\Lambda}_{1s*}^2 & 2\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s*} & \mathbf{0} \\ \mathbf{0} & -\bar{\mathbf{L}}_{s*} \end{bmatrix} \\ &= \tilde{\mathbf{L}}_{s*} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -2(\mathbf{I} + \Lambda_{1s*}^{-2})^{-1} \\ \mathbf{0} & \mathbf{I} & -2(\mathbf{I} + \bar{\Lambda}_{1s*}^2)^{-1} & \mathbf{0} \end{bmatrix}, \\ \tilde{\mathbf{L}}_{s*} &= \begin{bmatrix} (\mathbf{I} + \Lambda_{1s*}^{-2})(\Lambda_{1s*} - \Lambda_{1s*}^{-1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} + \bar{\Lambda}_{1s*}^2)(\bar{\Lambda}_{1s*}^{-1} - \bar{\Lambda}_{1s*})^{-1} \end{bmatrix}.\end{aligned}$$

Combining the above identities, we obtain

$$\hat{\mathbf{Z}}^{-1} \mathbf{m}^{-1} = \text{diag}(\mathbf{L}_{e*}, \mathbf{L}_{h*}, \tilde{\mathbf{L}}_{s*}) \left( \mathbf{I}_p, -2 \text{diag} \left( \Gamma_{e*}, \Lambda_{1h*} \Gamma_{h*}, \begin{bmatrix} \mathbf{0} & \tilde{\Gamma}_{s*} \\ \Gamma_{s*} & \mathbf{0} \end{bmatrix} \right) \right)$$

with  $\tilde{\Gamma}_{s*} = \mathbf{I} - \bar{\Gamma}_{1s*}$  and

$$(3.32) \quad \Gamma_{e*} = (\Lambda_{1e*} + \Lambda_{1e*}^{-1})^{-1}, \quad \Gamma_{h*} = (\Lambda_{1h*} + \Lambda_{1h*}^{-1})^{-1}, \quad \Gamma_{s*} = (\mathbf{I} + \bar{\Lambda}_{1s*}^2)^{-1}.$$

We define  $\tilde{\mathbf{B}}_j$  to be the  $j$ -th row of

$$(3.33) \quad \tilde{\mathbf{B}} := \mathbf{B}^{-1} \text{diag}(\mathbf{L}_{e*}, \mathbf{L}_{h*}, \tilde{\mathbf{L}}_{s*}).$$

With  $\mathbf{z}'_{s*} = (z_{p-s*+1}, \dots, z_p)$ , the defining equations of  $M$  are given by

$$z''_{p+j} = \{ \tilde{\mathbf{B}}_j \cdot (\mathbf{z}_{e*} - 2\Gamma_{e*} \bar{\mathbf{z}}_{e*}, \mathbf{z}_{h*} - 2\Gamma_{h*} \Lambda_{1h*} \bar{\mathbf{z}}_{h*}, \mathbf{z}_{s*} - 2\Gamma_{s*} \bar{\mathbf{z}}'_{s*}, \mathbf{z}'_{s*} - 2(\mathbf{I} - \bar{\Gamma}_{s*}) \bar{\mathbf{z}}_{s*}) \}^2.$$

Let us replace  $z_j$  with  $j \neq h$ ,  $z_h$  by  $iz_j$  and  $i\sqrt{\lambda_h} z_h$ , respectively for  $1 \leq j \leq p$ . Replace  $z_{p+j}$  by  $-z_{p+j}$ . In the new coordinates,  $M$  is given by

$$z''_{p+j} = \{ \hat{\mathbf{B}}_j \cdot (\mathbf{z}_{e*} + 2\Gamma_{e*} \bar{\mathbf{z}}_{e*}, \mathbf{z}_{h*} + 2\Gamma_{h*} \bar{\mathbf{z}}_{h*}, \mathbf{z}_{s*} + 2\tilde{\Gamma}_{s*} \bar{\mathbf{z}}'_{s*}, \mathbf{z}'_{s*} + 2\Gamma_{s*} \bar{\mathbf{z}}_{s*}) \}^2.$$

Explicitly, we have

$$(3.34) \quad Q_{\mathbf{B}, \gamma} : z_{p+j} = \left( \sum_{\ell=1}^{e_*+h_*} \hat{b}_{j\ell} (z_\ell + 2\gamma_\ell \bar{z}_\ell) + \sum_{s=e_*+h_*+1}^{p-s_*} \hat{b}_{js} (z_s + 2\gamma_{s+s_*} \bar{z}_{s+s_*}) + \hat{b}_{j(s+s_*)} (z_{s+s_*} + 2\gamma_s \bar{z}_s) \right)^2$$

for  $1 \leq j \leq p$ . Here

$$\gamma_{s+s_*} = 1 - \bar{\gamma}_s.$$

By (3.33), we also obtain the following identity

$$\hat{\mathbf{B}} = \mathbf{B}^{-1} \text{diag}(\mathbf{L}_{e*}, \mathbf{L}_{h*}, \tilde{\mathbf{L}}_{s*}) \text{diag}(\mathbf{I}_{e*}, \Lambda_{1h*}^{1/2}, \mathbf{I}_{2s*}).$$

The equivalence relation (3.27) on the set of non-singular matrices  $\mathbf{B}$  now takes the form

$$(3.35) \quad \hat{\mathbf{B}} = (\text{diag}_\nu \mathbf{d})^{-1} \hat{\mathbf{B}} \text{diag } \mathbf{a},$$

where  $\mathbf{a}$  satisfies (3.26) and  $\text{diag}_\nu \mathbf{d}$  is defined in (3.20).

Therefore, by Proposition 2.1 we obtain the following classification for the quadrics.

**Theorem 3.7.** *Let  $M$  be a quadratic submanifold defined by (2.1)-(2.2) with  $q^{-1}(0) = 0$ . Assume that the branched covering  $\pi_1$  has  $2^p$  deck transformations. Let  $T_1, T_2$  be the pair of Moser-Webster involutions of  $M$ . Suppose that  $S = T_1 T_2$  has  $2p$  distinct eigenvalues. Then  $M$  is holomorphically equivalent to (3.34) with  $\hat{\mathbf{B}} \in GL(p, \mathbf{C})$  being uniquely determined by the equivalence relation (3.35).*

When  $\hat{\mathbf{B}}$  is the identity, we obtain the product of 3 types of quadrics

$$(3.36) \quad \begin{aligned} \mathcal{Q}_{\gamma_e} : z_{p+e} &= (z_e + 2\gamma_e \bar{z}_e)^2; \\ \mathcal{Q}_{\gamma_h} : z_{p+h} &= (z_h + 2\gamma_h \bar{z}_h)^2; \\ \mathcal{Q}_{\gamma_s} : z_{p+s} &= (z_s + 2(1 - \bar{\gamma}_s) \bar{z}_{s+s_*})^2, \quad z_{p+s+s_*} = (z_{s+s_*} + 2\gamma_s \bar{z}_s)^2 \end{aligned}$$

with

$$\gamma_e = \frac{1}{\lambda_e + \lambda_e^{-1}}, \quad \gamma_h = \frac{1}{\lambda_h + \bar{\lambda}_h}, \quad \gamma_s = \frac{1}{1 + \bar{\lambda}_s^2}.$$

Note that  $\arg \lambda_s \in (0, \pi/2)$  and  $|\lambda_s| > 1$ . Thus

$$(3.37) \quad 0 < \gamma_e < 1/2, \quad \gamma_h > 1/2, \quad \gamma_s \in \{z \in \mathbf{C} : \text{Re } z > 1/2, \text{Im } z > 0\}.$$

**Remark 3.8.** By seeking simple formulae (3.29) for invariant functions  $Z^+$  of  $\{T_{1j}\}$  and (3.30) for invariant functions  $W^+$  of  $\{T_{2j}\} = \{\rho T_{1j} \rho\}$ , we have mismatched the indices so that  $W_{s+s_*}^+(\xi, \eta)$ , instead of  $W_s^+$ , is invariant by  $T_{2s}$ . In (3.36) for  $p = 2$  and  $h_* = e_* = 0$ , by interchanging  $(z_s, z_{p+s})$  with  $(z_{s+s_*}, z_{p+s+s_*})$  we get the quadric (1.2), an equivalent form of (3.36).

We define the following invariants.

**Definition 3.9.** We call  $\mathbf{\Gamma} = \text{diag}(\mathbf{\Gamma}_{e_*}, \mathbf{\Gamma}_{h_*}, \mathbf{\Gamma}_{s_*}, \mathbf{I}_{s_*} - \bar{\mathbf{\Gamma}}_{s_*})$ , given by formulae (3.32), the *Bishop invariants* of the quadrics. The equivalence classes  $\hat{\mathbf{B}}$  of non-singular matrices  $\mathbf{B}$  under the equivalence relation (3.27) are called the *extended Bishop invariants* for the quadrics.

Note that  $\mathbf{\Gamma}_{e_*}$  has diagonal elements in  $(0, 1/2)$ , and  $\mathbf{\Gamma}_{h_*}$  has diagonal elements in  $(1/2, \infty)$ , and  $\mathbf{\Gamma}_{s_*}$  has diagonal elements in  $(-\infty, 1/2) + i(0, \infty)$ .

We remark that  $Z_j^-$  is skew-invariant by  $T_{1i}$  for  $i \neq j$  and invariant by  $\tau_{1j}$ . Therefore, the square of a linear combination of  $Z_1^-, \dots, Z_p^-$  might not be invariant by all  $T_{1j}$ . This explains the presence of  $\mathbf{B}$  as invariants in the normal form.

It is worthy stating the following normal form for two families of linear holomorphic involutions which may not satisfy the reality condition.

**Proposition 3.10.** *Let  $\mathcal{T}_i = \{T_{i1}, \dots, T_{ip}\}$ ,  $i = 1, 2$  be two families of distinct and commuting linear holomorphic involutions on  $\mathbf{C}^n$ . Let  $T_i = T_{i1} \cdots T_{ip}$ . Suppose that for each  $i$ ,  $\text{Fix}(T_{i1}), \dots, \text{Fix}(T_{ip})$  are hyperplanes intersecting transversally. Suppose that  $T_1, T_2$  satisfy*



(3.2) and  $S = T_1 T_2$  has  $2p$  distinct eigenvalues. In suitable linear coordinates, the matrices of  $T_i, S$  are

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_i \\ \mathbf{\Lambda}_i^{-1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}$$

with  $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^{-1}$  being diagonal matrix whose entries do not contain  $\pm 1, \pm i$ . The  $\mathbf{\Lambda}_1^2$  is uniquely determined up to a permutation in diagonal entries. Moreover, the matrices of  $T_{ij}$  are

$$(3.38) \quad \mathbf{T}_{ij} = \mathbf{E}_{\mathbf{\Lambda}_i}(\mathbf{B}_i)_* \mathbf{Z}_j(\mathbf{B}_i)_*^{-1} \mathbf{E}_{\mathbf{\Lambda}_i}^{-1}$$

for some non-singular complex matrices  $\mathbf{B}_1, \mathbf{B}_2$  uniquely determined by the equivalence relation

$$(3.39) \quad (\mathbf{B}_1, \mathbf{B}_2) \sim (\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2) := ((\text{diag } \mathbf{a})^{-1} \mathbf{B}_1 \text{diag}_{\nu_1} \mathbf{d}_1, (\text{diag } \mathbf{a})^{-1} \mathbf{B}_2 \text{diag}_{\nu_2} \mathbf{d}_2),$$

where  $\text{diag}_{\nu_1} \mathbf{d}_1, \text{diag}_{\nu_2} \mathbf{d}_2$  are defined as in (3.20), and  $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$  is a non-singular diagonal complex matrix representing the linear transformation  $\varphi$  such that

$$\varphi^{-1} T_{ij} \varphi = \tilde{T}_{i\nu_i(j)}, \quad i = 1, 2, j = 1, \dots, p.$$

Here  $\tilde{T}_i$  is the family of the involutions associated to the matrices  $\tilde{\mathbf{B}}_i$ , and  $\mathbf{E}_{\mathbf{\Lambda}_i}$  and  $\mathbf{B}_*$  are defined by (3.22)-(3.23).

*Proof.* Let  $\kappa$  be an eigenvalue of  $S$  with (non-zero) eigenvector  $u$ . Since  $T_i S T_i = S^{-1}$ . Then  $S(T_i(u)) = \kappa^{-1} T_i(u)$ . This shows that  $\kappa^{-1}$  is also an eigenvalue of  $S$ . By Lemma 3.1, 1 and  $-1$  are not eigenvalues of  $S$ . Thus, we can list the eigenvalues of  $S$  as  $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$ . Let  $u_j$  be an eigenvector of  $S$  with eigenvalue  $\mu_j$ . Fix  $\lambda_j$  such that  $\lambda_j^2 = \mu_j$ . Then  $v_j := \lambda_j T_1(u_j)$  is an eigenvector of  $S$  with eigenvalue  $\mu_j^{-1}$ . The  $\sum \xi_j u_j + \eta_j v_j$  defines a coordinate system on  $\mathbf{C}^n$  such that  $T_i, S$  have the above matrices  $\mathbf{\Lambda}_i$  and  $\mathbf{S}$ , respectively. By (3.21) and (3.23),  $T_{ij}$  can be expressed in (3.38), where each  $\mathbf{B}_i$  is uniquely determined up to  $\mathbf{B}_i \text{diag } \mathbf{d}_i$ . Suppose that  $\{\tilde{T}_{1j}\}, \{\tilde{T}_{2j}\}$  are another pair of families of linear involutions of which the corresponding matrices are  $\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2$ . If there is a linear change of coordinates  $\varphi$  such that  $\varphi^{-1} T_{ij} \varphi = \tilde{T}_{i\nu_i(j)}$ , then in the matrix  $\mathbf{R}$  of  $\varphi$ , we obtain (3.39); see a similar computation for (3.27) by using (3.28). Conversely, (3.28) implies that the corresponding pairs of families of involutions are equivalent.  $\square$

Finally, we conclude the section with examples of quadratic manifolds of maximum deck transformations for which the corresponding  $\sigma$  is not diagonalizable.

**Example 3.11.** Let  $\mathbf{K}$  be a  $p \times p$  invertible matrix. Let  $T_1, \rho, T_2 = \rho T_1 \rho, S$  have matrices

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K}^{-1} & \mathbf{0} \end{pmatrix}, \quad \rho = \begin{pmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} \mathbf{0} & \overline{\mathbf{K}}^{-1} \\ \overline{\mathbf{K}} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{K} \overline{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{-1} \overline{\mathbf{K}}^{-1} \end{pmatrix}.$$

One can verify that the sets of fixed points of  $T_1, T_2$  intersect transversally if

$$(3.40) \quad \det(\mathbf{K} - \overline{\mathbf{K}}^{-1}) \neq 0.$$

We can decompose  $T_1 = T_{11} \cdots T_{1p}$  where  $T_{11}, \dots, T_{1p}$  are commuting involutions and each of them fixes a hyperplane by using

$$\begin{pmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K}^{-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1}.$$

In coordinates, we have  $T_1: (\xi, \eta)^t \rightarrow \mathbf{T}_1(\xi, \eta)^t$ . Thus the linear invariant functions of  $\{T_{11}, \dots, T_{1p}\}$  are precisely generated by linear invariant functions of  $T_1$ , and they are linear combinations of the entries of the column vector  $\xi^t + \mathbf{K}\eta^t$ . On the other hand, the linear invariant functions of  $\{T_{21}, \dots, T_{2p}\}$  are linear combinations of the entries of the vector  $\xi^t + \overline{\mathbf{K}}^{-1}\eta^t$ . The two sets of entries are linearly independent functions; indeed if there are row vectors  $\mathbf{a}, \mathbf{b}$  such that

$$\mathbf{a}(\xi^t + \mathbf{K}\eta^t) + \mathbf{b}(\xi^t + \overline{\mathbf{K}}^{-1}\eta^t) = 0$$

then  $\mathbf{a} = \mathbf{b}$  and  $\mathbf{a}(\mathbf{K} - \overline{\mathbf{K}}^{-1}) = \mathbf{0}$ . Thus  $\mathbf{a} = \mathbf{0}$  if (3.40) holds. Thus condition (3.40) also implies (3.1)-(3.2). By Proposition 2.1, the family of  $\{T_{11}, \dots, T_{1p}, \rho\}$ , in particular the matrix  $S$ , can be realized by a quadratic manifold.

For a more explicit example, let  $\mathbf{J}_p$  be the  $p \times p$  Jordan matrix with entries 1 or 0. Then  $\mathbf{K} = \lambda \mathbf{J}_p$  satisfies (3.40) if  $\lambda$  is positive and  $\lambda \neq 1$ , as  $\overline{\mathbf{K}}^{-1} = \overline{\lambda}^{-1} \mathbf{J}_p^{-1}$ . For another example, set

$$\mathbf{K}_\lambda = \begin{pmatrix} \mathbf{0} & \lambda \mathbf{J}_q \\ \overline{\lambda} \mathbf{J}_q & \mathbf{0} \end{pmatrix}, \quad \mathbf{K}_\lambda \overline{\mathbf{K}}_\lambda = \begin{pmatrix} \lambda^2 \mathbf{J}_q^2 & \mathbf{0} \\ \mathbf{0} & \overline{\lambda}^2 \mathbf{J}_q^2 \end{pmatrix}$$

with  $q = p/2$  and  $p$  even. If  $\lambda \in \mathbf{C}$  and  $\lambda \neq 0, \pm 1$  then  $K$  satisfies (3.40) as

$$\overline{\mathbf{K}}_\lambda^{-1} = \begin{pmatrix} \mathbf{0} & \lambda^{-1} \mathbf{J}_q^{-1} \\ \overline{\lambda}^{-1} \mathbf{J}_q^{-1} & \mathbf{0} \end{pmatrix}.$$

When  $\lambda = 1$  and  $q = 2$ , we obtain  $\mathbf{S}$  in (2.11) if the above  $\mathbf{J}_2$  is replaced by  $\mathbf{J}_2^{1/2}$ , the Jordan matrix with eigenvalue 1 and off-diagonal entries  $1/2$ .

#### 4. FORMAL DECK TRANSFORMATIONS AND CENTRALIZERS

In section 2 we describe the equivalence of the classification of real analytic submanifolds  $M$  that admit the maximum number of deck transformations and the classification of the families of involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  that satisfy some mild conditions (see Proposition 2.1). To classify the families of involutions and to find their normal forms, we will also study the centralizers of various linear maps to deal with resonance. This is relevant as the normal form of  $\sigma$  will belong to the centralizer of its linear part and any further normalization will also be performed by transformations that are in the centralizer.

In this subsection, we describe centralizers regarding  $\hat{S}, \hat{T}_1$  and  $\hat{\mathcal{T}}_1$ . We will also describe the complement sets of the centralizers, i.e. the sets of mappings which satisfy suitable normalizing conditions. Roughly speaking, our normal forms are in the centralizers and coordinate transformations that achieve the normal forms are normalized, while an arbitrary formal transformation admits a unique decomposition of a mapping in a centralizer and a mapping in the complement of the centralizer.

Recall that

$$(4.1) \quad \hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p,$$

$$(4.2) \quad \hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad 1 \leq j \leq p$$

with  $\mu_j = \lambda_{1j}^2$  and  $\lambda_{2j}^{-1} = \lambda_{1j} = \lambda_j$ .

**Definition 4.1.** Let  $\mathcal{F}$  be a family of formal mappings on  $\mathbf{C}^n$  fixing the origin. Let  $\mathcal{C}(\mathcal{F})$  be the *centralizer* of  $\mathcal{F}$ , i.e. the set of formal holomorphic mappings  $g$  that fix the origin and commute with each element  $f$  of  $\mathcal{F}$ , i.e.,  $f \circ g = g \circ f$ .

Note that we do not require that elements in  $\mathcal{C}(\mathcal{F})$  be invertible or convergent.

We first compute the centralizers.

**Lemma 4.2.** Let  $\hat{S}$  be given by (4.1) with  $\mu_1, \dots, \mu_p$  being non-resonant. Then  $\mathcal{C}(\hat{S})$  consists of mappings of the form

$$(4.3) \quad \psi: \xi'_j = a_j(\xi\eta)\xi_j, \quad \eta'_j = b_j(\xi\eta)\eta_j, \quad 1 \leq j \leq p.$$

Let  $\tau_1, \tau_2$  be formal holomorphic involutions such that  $\hat{S} = \tau_1 \tau_2$ . Then

$$\tau_i: \xi'_j = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta'_j = \Lambda_{ij}^{-1}(\xi\eta)\xi_j, \quad 1 \leq j \leq p$$

with  $\Lambda_{1j}\Lambda_{2j}^{-1} = \mu_j$ . The centralizer of  $\{\hat{T}_1, \hat{T}_2\}$  consists of the above transformations satisfying

$$(4.4) \quad b_j = a_j, \quad 1 \leq j \leq p.$$

*Proof.* Let  $e_j = (0, \dots, 1, \dots, 0) \in \mathbf{N}^p$ , where 1 is at the  $j$ th place. Let  $\psi$  be given by

$$\xi'_j = \sum a_{j,PQ} \xi^P \eta^Q, \quad \eta'_j = \sum b_{j,PQ} \xi^P \eta^Q.$$

By the non-resonance condition, it is straightforward that if  $\psi\hat{S} = \hat{S}\psi$ , then  $a_{j,PQ} = b_{j,QP} = 0$  if  $P - Q \neq e_j$ . Note that  $\hat{S}^{-1} = T_0 \hat{S} T_0$  for  $T_0: (\xi, \eta) \rightarrow (\eta, \xi)$ . Thus  $\tau_1 T_0$  commutes with  $\hat{S}$ . So  $\tau_1 T_0$  has the form (4.3) in which we rename  $a_j, b_j$  by  $\Lambda_{1j}, \tilde{\Lambda}_{1j}$ , respectively. Now  $\tau_1^2 = \text{I}$  implies that

$$\Lambda_{1j}((\Lambda_{11}\tilde{\Lambda}_{11})(\zeta)\zeta_1, \dots, (\Lambda_{1p}\tilde{\Lambda}_{1p})(\zeta)\zeta_p)\tilde{\Lambda}_{1j}(\zeta) = 1, \quad 1 \leq j \leq p.$$

Then  $\Lambda_{1j}(0)\tilde{\Lambda}_{1j}(0) = 1$ . Applying induction on  $d$ , we verify that for all  $j$

$$\Lambda_{1j}(\zeta)\tilde{\Lambda}_{1j}(\zeta) = 1 + O(|\zeta|^d), \quad d > 1.$$

Having found the formula for  $\tau_1 T_0$ , we obtain the desired formula of  $\tau_1$  via composition  $(\tau_1 T_0)T_0$ .  $\square$

Let  $\mathbf{D}_1 := \text{diag}(\mu_{11}, \dots, \mu_{1n}), \dots, \mathbf{D}_\ell := \text{diag}(\mu_{\ell 1}, \dots, \mu_{\ell n})$  be diagonal invertible matrices of  $\mathbf{C}^n$ . Let us set  $D := \{\mathbf{D}_i z\}_{i=1, \dots, \ell}$ .

**Definition 4.3.** Let  $F$  be a formal mapping of  $\mathbf{C}^n$  that is tangent to the identity.

(i) Let  $n = 2p$ .  $F$  is *normalized* with respect to  $\hat{S}$ , if  $F = (f, g)$  is tangent to the identity and  $F$  contains no resonant terms, i.e.

$$f_{j, (A+e_j)A} = 0 = g_{j, A(A+e_j)}, \quad |A| > 1.$$

(ii) Let  $n = 2p$ .  $F$  is *normalized* with respect to  $\{\hat{T}_1, \hat{T}_2\}$ , if  $F = (f, g)$  is tangent to the identity and

$$f_{j, (A+e_j)A} = -g_{j, A(A+e_j)}, \quad |A| > 1.$$

(iii)  $F$  is *normalized* with respect to  $D$  if it does not have components along the centralizer of  $D$ , i.e. for each  $Q$  with  $|Q| \geq 2$ ,

$$f_{j, Q} = 0, \quad \text{if } \mu_i^Q = \mu_{ij} \text{ for all } i.$$

Let  $\mathcal{C}^c(\hat{S})$  (resp.  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ ,  $\mathcal{C}^c(D)$ ) denote the set of formal mappings normalized with respect to  $\hat{S}$  (resp.  $\{\hat{T}_1, \hat{T}_2\}$ , the family  $D$ ). For convenience, we let  $\mathcal{C}_2^c(\hat{S})$  (resp.  $\mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$ ,  $\mathcal{C}_2^c(D)$ ) denote the set of formal mappings  $F - I$  with  $F \in \mathcal{C}^c(\hat{S})$  (resp.  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ ,  $\mathcal{C}^c(D)$ ).

Recall that for  $j = 1, \dots, p$ , we define

$$Z_j: \xi' = \xi, \quad \eta'_k = \eta_k, \quad k \neq j, \quad \eta'_j = -\eta_j.$$

We have seen in section 3 how invariant functions of  $Z_j$  play a role in constructing normal form of quadrics. In section 7, we will also need a centralizer for non linear maps (see Lemma 7.2) to obtain normal forms for two families of involutions. Therefore, let us first recall the following lemma on the centralizer of  $Z_1, \dots, Z_p$ , which is a special case of [GS15, Lemma 4.7].

**Lemma 4.4.** *The centralizer,  $\mathcal{C}(Z_1, \dots, Z_p)$ , consists of formal mappings*

$$(\xi, \eta) \rightarrow (U(\xi, \eta), \eta_1 V_1(\xi, \eta), \dots, \eta_p V_p(\xi, \eta))$$

*such that  $U(\xi, \eta), V(\xi, \eta)$  are even in each  $\eta_j$ . Let  $\mathcal{C}^c(Z_1, \dots, Z_p)$  denote the set of mappings  $I + (U, V)$  which are tangent to the identity such that*

$$U_{j, PQ} = V_{j, P(e_j + Q')} = 0, \quad Q, Q' \in 2\mathbf{N}^p, \quad |P| + |Q| > 1, \quad |P| + |Q'| > 1.$$

*Let  $\psi$  be a mapping that is tangent to the identity. There exist unique  $\psi_0 \in \mathcal{C}(Z_1, \dots, Z_p)$  and  $\psi_1 \in \mathcal{C}^c(Z_1, \dots, Z_p)$  such that  $\psi = \psi_1 \psi_0^{-1}$ . Moreover, if  $\psi$  is convergent, then  $\psi_0$  and  $\psi_1$  are convergent.*

Analogously, for any formal mapping  $\psi$  that is tangent to the identity, there is a unique decomposition  $\psi = \psi_1 \psi_0^{-1}$  with  $\psi_1 \in \mathcal{C}^c(\hat{S})$  and  $\psi_0 \in \mathcal{C}(\hat{S})$ . If  $\psi$  is convergent, then  $\psi_0, \psi_1$  are convergent. Let  $F = (F_1, \dots, F_n): \mathbf{C}^n \rightarrow \mathbf{C}^n$  be a formal mapping. Define a formal mapping  $F_{sym}: \mathbf{C}^n \rightarrow \mathbf{C}^n$  by

$$(F_{sym})_{i, P} = \max_{1 \leq j \leq n, \nu \in S_n} |\{g_j \circ \nu\}_P|,$$

where  $S_n$  is the set of permutations  $\nu$  of coordinates  $z_i \rightarrow z_{\nu(i)}$ . Let us recall the following lemma from [GS15, Lemma 4.3].

**Lemma 4.5.** *Let  $\hat{\mathcal{H}}$  be a real subspace of  $(\widehat{\mathfrak{M}}_n^2)^n$ . Let  $\pi: (\widehat{\mathfrak{M}}_n^2)^n \rightarrow \hat{\mathcal{H}}$  be a  $\mathbf{R}$  linear projection (i.e.  $\pi^2 = \pi$ ) that preserves the degrees of the mappings and let  $\hat{\mathcal{G}} := (I - \pi)(\widehat{\mathfrak{M}}_n^2)^n$ . Suppose that there is a positive constant  $C$  such that  $\pi(E) \prec C E_{sym}$  for any  $E \in (\widehat{\mathfrak{M}}_n^2)^n$ . Let  $F$  be a formal map tangent to the identity. There exists a unique decomposition  $F = H G^{-1}$  with  $G - I \in \hat{\mathcal{G}}$  and  $H - I \in \hat{\mathcal{H}}$ . If  $F$  is convergent, then  $G$  and  $H$  are also convergent.*

**Lemma 4.6.** *Let  $\psi$  be a mapping that is tangent to the identity. There exist unique  $\psi_0 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$  and  $\psi_1 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  such that  $\psi = \psi_1 \psi_0^{-1}$ . Moreover, if  $\psi$  is convergent, then  $\psi_0$  and  $\psi_1$  are convergent.*

*Proof.* Let  $\hat{\mathcal{G}} = \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$  and  $\hat{\mathcal{H}} = \mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$ . We need to find a  $\mathbf{R}$ -linear projection such that  $\hat{\mathcal{H}} = \pi(\widehat{\mathfrak{M}}_n^2)^n$ ,  $\hat{\mathcal{G}} = (\mathbf{I} - \pi)(\widehat{\mathfrak{M}}_n^2)^n$ , and  $\pi(E) \prec CE_{sym}$ . Note that  $g \in \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$  and  $h \in \mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$  are determined by conditions

$$\begin{aligned} g_{j,(\gamma+e_j)\gamma} &= g_{(j+p),\gamma(\gamma+e_j)}, & h_{j,(\gamma+e_j)\gamma} &= -h_{(j+p),\gamma(\gamma+e_j)}, & 1 \leq j \leq p, \\ g_{j,PQ} &= g_{(j+p),QP} = 0, & P - Q &\neq e_j. \end{aligned}$$

Thus, if  $h - g = K$ , we determine  $g$  uniquely by combining the above identities with

$$\begin{aligned} g_{j,(\gamma+e_j)\gamma} &= \frac{-1}{2} \{K_{j,(\gamma+e_j)\gamma} + K_{(j+p),\gamma(\gamma+e_j)}\}, \\ h_{j,(\gamma+e_j)\gamma} &= \frac{1}{2} \{K_{j,(\gamma+e_j)\gamma} - K_{(j+p),\gamma(\gamma+e_j)}\} \end{aligned}$$

for  $1 \leq j \leq p$ . For the remaining coefficients of  $h$ , set  $h_{i,PQ} = K_{i,PQ}$ . Therefore,  $\pi(K) := h \prec K_{sym}$  and the lemma follows from Lemma 4.5.  $\square$

## 5. FORMAL NORMAL FORMS OF THE REVERSIBLE MAP $\sigma$

Let us first describe our plans to derive the normal forms of  $M$ . We would like to show that two families of involutions  $\{\tau_{1j}, \tau_{2j}, \rho\}$  and  $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$  are holomorphically equivalent, if their corresponding normal forms are equivalent under a much smaller set of changes of coordinates. Ideally, we would like to conclude that  $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$  are holomorphically equivalent if and only if their corresponding normal forms are the same, or if they are the same under a change of coordinates with finitely many parameters. For instance the Moser-Webster normal form for real analytic surfaces ( $p = 1$ ) with non-vanishing elliptic Bishop invariant falls into the former situation, while the Chern-Moser theory [CM74] for real analytic hypersurfaces with non-degenerate Levi-form is an example for the latter. Such a normal form will tell us if the real manifolds have infinitely many invariants or not. One of our goals is to understand if the normal form so achieved can be realized by a convergent normalizing transformation. We will see soon that we can achieve our last goal under some assumptions on the family of involutions. Alternatively and perhaps for simplicity of the normal form theory, we would like to seek normal forms which are dynamically or geometrically significant.

Recall that for each real analytic manifold that has  $2^p$ , the maximum number of, commuting deck transformations  $\{\tau_{1j}\}$ , we have found a unique set of generators  $\tau_{11}, \dots, \tau_{1p}$  so that each  $\text{Fix}(\tau_{1j})$  has codimension 1. More importantly  $\tau_1 = \tau_{11} \cdots \tau_{1p}$  is the unique deck transformation of which the set of fixed points has dimension  $p$ . Let  $\tau_2 = \rho \tau_1 \rho$  and  $\sigma = \tau_1 \tau_2$ . To normalize  $\{\tau_{1j}, \tau_{2j}, \rho\}$ , we will choose  $\rho$  to be the standard anti-holomorphic involution determined by the linear parts of  $\sigma$ . Then we normalize  $\sigma = \tau_1 \tau_2$  under formal mapping commuting with  $\rho$ . This will determine a normal form for  $\{\tau_1^*, \tau_2^*, \rho\}$ . This part of normalization is analogous to the Moser-Webster normalization. When  $p = 1$ , Moser and Webster obtained a unique normal form by a simple argument. However, this last step of simple normalization is not available when  $p > 1$ . By assuming  $\log \hat{M}$  associated to  $\hat{\sigma}$

is tangent to the identity, we will obtain a unique formal normal form  $\hat{\sigma}, \hat{\tau}_1, \hat{\tau}_2$  for  $\sigma, \tau_1, \tau_2$ . Next, we need to construct the normal form for the families of involutions. We first ignore the reality condition, by finding  $\Phi$  which transforms  $\{\tau_{1j}\}$  into a set of involutions  $\{\hat{\tau}_{1j}\}$  which is decomposed canonically according to  $\hat{\tau}_1$ . This allows us to express  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  via  $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$ , as in the classification of the families of linear involutions. Finally, we further normalize  $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$  to get our normal form.

**Definition 5.1.** Throughout this section and next, we denote  $\{h\}_d$  the set of coefficients of  $h_P$  with  $|P| \leq d$  if  $h(x)$  is a map or function in  $x$  as power series. We denote by  $\mathcal{A}_P(t), \mathcal{A}(y; t)$ , etc., a universal *polynomial* whose coefficients and degree depend on a multiindex. The variables in these polynomials will involve a collection of Taylor coefficients of various mappings. The collection will also depend on  $|P|$ . As such dependency (or independency to coefficients of higher degrees) is crucial to our computation, we will remind the reader the dependency when emphasis is necessary.

For instance, let us take two formal mappings  $F, G$  from  $\mathbf{C}^n$  into itself. Suppose that  $F = I + f$  with  $f(x) = O(|x|^2)$  and  $G = LG + g$  with  $g(x) = O(|x|^2)$  and  $LG$  being linear. For  $P \in \mathbf{N}^n$  with  $|P| > 1$ , we can express

$$(5.1) \quad (F^{-1})_P = -f_P + \mathcal{F}_P(\{f\}_{|P|-1}),$$

$$(5.2) \quad (G \circ F)_P = g_P + ((LG) \circ f)_P + \mathcal{G}_P(LG; \{f, g\}_{|P|-1}),$$

$$(5.3) \quad (F^{-1} \circ G \circ F)_P = g_P - (f \circ (LG))_P + ((LG) \circ f)_P + \mathcal{H}_P(LG; \{f, g\}_{|P|-1}).$$

**5.1. Formal normal forms of pair of involutions  $\{\tau_1, \tau_2\}$ .** We first find a normal form for  $\sigma$  in  $\mathcal{C}(S)$ .

**Proposition 5.2.** *Let  $\sigma$  be a holomorphic map. Suppose that  $\sigma$  has a non-resonant linear part*

$$\hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p.$$

*Then there exists a unique normalized formal map  $\Psi \in \mathcal{C}^c(\hat{S})$  such that  $\sigma^* = \Psi^{-1} \sigma \Psi \in \mathcal{C}(\hat{S})$ . Moreover,  $\tilde{\sigma} = \psi_0^{-1} \sigma^* \psi_0 \in \mathcal{C}(\hat{S})$ , if and only if  $\psi_0 \in \mathcal{C}(\hat{S})$  and it is invertible. Let*

$$\begin{aligned} \sigma^*: \xi'_j &= M_j(\xi \eta) \xi_j, & \eta'_j &= N_j(\xi \eta) \eta_j, \\ \tilde{\sigma}: \xi'_j &= \tilde{M}_j(\xi \eta) \xi_j, & \eta'_j &= \tilde{N}_j(\xi \eta) \eta_j, \\ \psi_0: \xi'_j &= a_j(\xi \eta) \xi_j, & \eta'_j &= b_j(\xi \eta) \eta_j. \end{aligned}$$

(i) *Assume that  $\tau_1, \tau_2$  are holomorphic involutions and  $\sigma = \tau_1 \tau_2$ . Then  $\sigma^* = \tau_1^* \tau_2^*$  with*

$$(5.4) \quad \begin{aligned} \tau_i^* &= \Psi^{-1} \tau_i \Psi: \xi'_j = \Lambda_{ij}(\xi \eta) \eta_j, & \eta'_j &= \Lambda_{ij}^{-1}(\xi \eta) \xi_j; \\ N_j &= M_j^{-1}, & M_j &= \Lambda_{1j} \Lambda_{2j}^{-1}. \end{aligned}$$

*Let the linear part of  $\tau_i$  be given by*

$$\hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j.$$

Suppose that  $\lambda_{2j}^{-1} = \lambda_{1j}$ . There exists a unique  $\psi_0 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  such that

$$(5.5) \quad \begin{aligned} \tilde{\tau}_i &= \psi_0^{-1} \tau_i^* \psi_0: \xi'_j = \tilde{\Lambda}_{ij}(\xi\eta)\eta_j, \quad \eta'_j = \tilde{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j; \\ \tilde{M}_j &= \tilde{\Lambda}_{1j}^2 = \tilde{N}_j^{-1}, \quad \tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}. \end{aligned}$$

Let  $\psi_1$  be a formal biholomorphic map. Then  $\{\psi_1^{-1}\tilde{\tau}_1\psi_1, \psi_1^{-1}\tilde{\tau}_2\psi_1\}$  has the same form as of  $\{\tilde{\tau}_1, \tilde{\tau}_2\}$  if and only if  $\psi_1 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$ ; moreover,  $\tilde{\Lambda}_{ij}(\xi\eta)$ ,  $\tilde{M}_j(\xi\eta)$  are transformed into

$$(5.6) \quad \tilde{\Lambda}_{ij} \circ \tilde{\psi}_1, \quad \tilde{M}_j \circ \tilde{\psi}_1.$$

Here  $\tilde{\psi}_1(\zeta) = (\text{diag } c(\zeta))^2 \zeta$  and  $\psi_1(\xi, \eta) = ((\text{diag } c(\xi\eta))\xi, (\text{diag } c(\xi\eta))\eta)$ .

(ii) Assume further that  $\tau_2 = \rho\tau_1\rho$ , where  $\rho$  is defined by (3.8). Let

$$\rho_z: \zeta_j \rightarrow \bar{\zeta}_j, \quad 1 \leq j \leq e_* + h_*; \quad \zeta_s \rightarrow \bar{\zeta}_{s+s_*}, \quad e_* + h_* < s \leq p - s_*.$$

Then  $\rho\Psi = \Psi\rho$ ,  $\tau_2^* = \rho\tau_1^*\rho$ , and  $(\sigma^*)^{-1} = \rho\sigma^*\rho$ . The last two identities are equivalent to

$$(5.7) \quad \Lambda_{2e}^{-1} = \overline{\Lambda_{1e} \circ \rho_z}, \quad \overline{M_e \circ \rho_z} = M_e, \quad 1 \leq e \leq e_*;$$

$$(5.8) \quad \Lambda_{2h} = \overline{\Lambda_{1h} \circ \rho_z}, \quad \overline{M_h \circ \rho_z} = M_h^{-1}, \quad e_* < h \leq h_* + e_*;$$

$$(5.9) \quad \Lambda_{2(s)} = \overline{\Lambda_{1(s_*+s)} \circ \rho_z},$$

$$(5.10) \quad \Lambda_{2(s_*+s)} = \overline{\Lambda_{1s} \circ \rho_z}, \quad \overline{M_s^{-1} \circ \rho_z} = M_{s_*+s}, \quad h_* + e_* < s \leq p - s_*.$$

Let  $\psi_0$  and  $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$  be as in (i). Then  $\rho\psi_0 = \psi_0\rho$ , and  $\hat{\tau}_1, \hat{\tau}_2$  satisfy

$$(5.11) \quad \tilde{\Lambda}_{ie} = \overline{\tilde{\Lambda}_{ie} \circ \rho_z}, \quad \tilde{\Lambda}_{ih}^{-1} = \overline{\tilde{\Lambda}_{ih} \circ \rho_z}, \quad \tilde{\Lambda}_{is+s_*} = \overline{\tilde{\Lambda}_{is}^{-1} \circ \rho_z}.$$

*Proof.* We will use the Taylor formula

$$f(x+y) = f(x) + \sum_{k=1}^m \frac{1}{k!} D_k f(x; y) + R_{m+1} f(x; y)$$

with  $D_k f(x; y) = \{\partial_t^k f(x+ty)\}|_{t=0}$  and

$$(5.12) \quad R_{m+1} f(x; y) = (m+1) \int_0^1 (1-t)^m \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha dt.$$

Set  $D = D_1$ . Let  $\sigma$  be given by

$$\xi'_j = M_j^0(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = N_j^0(\xi\eta)\eta_j + g_j(\xi, \eta)$$

with

$$(5.13) \quad (f, g) \in \mathcal{C}_2^c(\hat{S}), \quad \text{ord}(f, g) = d \geq 2.$$

We need to find  $\Phi \in \mathcal{C}^c(S)$  such that  $\Psi^{-1}\sigma\Psi = \sigma^*$  is given by

$$\xi'_j = M_j(\xi\eta)\xi_j, \quad \eta'_j = N_j(\xi\eta)\eta_j.$$

By definition,  $\Psi$  has the form

$$\xi'_j = \xi_j + U_j(\xi, \eta), \quad \eta'_j = \eta_j + V_j(\xi, \eta), \quad U_{j, (P+e_j)P} = V_{j, P(P+e_j)} = 0.$$

The components of  $\Psi\sigma^*$  are

$$(5.14) \quad \xi'_j = M_j(\xi\eta)\xi_j + U_j(M(\xi\eta)\xi, N(\xi\eta)\eta),$$

$$(5.15) \quad \eta'_j = N_j(\xi\eta)\eta_j + V_j(M(\xi\eta)\xi, N(\xi\eta)\eta).$$

To derive the normal form, we only need Taylor theorem in order one. This can also demonstrate small divisors in the normalizing transformation; however, one cannot see the small divisors in the normal forms. Later we will show the existence of divergent normal forms. This requires us to use Taylor formula whose remainder has order two. By the Taylor theorem, we write the components of  $\sigma\Psi$  as

$$(5.16) \quad \begin{aligned} \xi'_j &= (M_j^0(\xi\eta) + DM_j^0(\xi\eta)(\eta U + \xi V + UV))(\xi_j + U_j) \\ &\quad + f_j(\xi, \eta) + Df_j(\xi, \eta)(U, V) + A_j(\xi, \eta), \end{aligned}$$

$$(5.17) \quad \begin{aligned} \eta'_j &= (N_j^0(\xi\eta) + DN_j^0(\xi\eta)(\eta U + \xi V + UV))(\eta_j + V_j) \\ &\quad + g_j(\xi, \eta) + Dg_j(\xi, \eta)(U, V) + B_j(\xi, \eta). \end{aligned}$$

Recall our notation that  $UV = (U_1(\xi, \eta)V_1(\xi, \eta), \dots, U_p(\xi, \eta)V_p(\xi, \eta))$ . The second order remainders are

$$(5.18) \quad A_j(\xi, \eta) = R_2M_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) + R_2f_j(\xi, \eta; U, V),$$

$$(5.19) \quad B_j(\xi, \eta) = R_2N_j^0(\xi\eta; \xi U + \eta V + UV)(\eta_j + V_j) + R_2g_j(\xi, \eta; U, V).$$

Note that the remainder  $R_2M^0$  is independent of the linear part of  $M^0$ . Thus

$$R_2M_j^0 = R_2(M_j^0 - LM_j^0), \quad R_2N_j^0 = R_2(N_j^0 - LN_j^0).$$

Let us calculate the largest degrees  $w, d'$  of coefficients of  $M^0 - LM^0, (U, V, f, g)$  on which  $A_{j,PQ}$  depend. It is easy to see that  $d' \geq d \geq 2$  and  $w \geq 2$ . We have

$$\begin{aligned} 2(w-2) + 2(d+1) + 1 &\leq |P| + |Q|; \\ 3 + d + d' &\leq |P| + |Q| \quad \text{or} \quad 2d + d' - 2 \leq |P| + |Q|, \end{aligned}$$

where the first two inequalities are obtained from the first term on the right-hand side of (5.18) and its second term yields the last inequality. Thus, we have crude bounds

$$w \leq \frac{|P| + |Q| + 1 - 2d}{2}, \quad d' \leq |P| + |Q| - d.$$

Analogously, we can estimate the degrees of coefficients of  $N^0$ . We obtain

$$(5.20) \quad A_{j,PQ} = \mathcal{A}_{j,PQ}(\{M^0 - LM^0\}_{\frac{|P|+|Q|+1-2d}{2}}; \{f, U, V\}_{|P|+|Q|-d}),$$

$$(5.21) \quad B_{j,QP} = \mathcal{B}_{j,QP}(\{N^0 - LN^0\}_{\frac{|P|+|Q|+1-2d}{2}}; \{g, U, V\}_{|P|+|Q|-d}).$$

Recall our notation that  $\{f, U, V\}_d$  is the set of coefficients of  $f_{PQ}, U_{PQ}, V_{PQ}$  with  $|P|+|Q| \leq d$ . Here  $\mathcal{A}_{j,PQ}(t'; t''), \mathcal{B}_{j,QP}(t'; t'')$  are polynomials of which each has coefficients that depend only on  $j, P, Q$  and they vanish at  $t'' = 0$ .

To finish the proof of the proposition, we will not need the explicit expressions involving  $DM_j^0, DN_j^0, Df_j, Dg_j$ . We will use these derivatives in the proof of Lemma 6.1. So we derive these expression in this proof too.



We apply the projection (5.14)-(5.15) and (5.16)-(5.17) onto  $\mathcal{C}_2^c(S)$ , via monomials in each component of both sides of the identities. The images of the mappings

$$\begin{aligned}(\xi, \eta) &\mapsto (U(M(\xi\eta)\xi, N(\xi\eta)\eta), V(M(\xi\eta)\xi, N(\xi\eta)\eta)), \\(\xi, \eta) &\mapsto (M^0(\xi\eta)U(\xi, \eta), N^0(\xi\eta)V(\xi, \eta))\end{aligned}$$

under the projection are 0. We obtain from (5.14)-(5.17) and (5.18)-(5.19) that  $d_0 = d$ . Next, we project (5.14)-(5.15) and (5.16)-(5.17) onto  $\mathcal{C}_2(\hat{S})$ , via monomials in each component of both sides of the identities. Using (5.13) and (5.20) we obtain

$$(5.22) \quad M_{j,P} = M_{j,P}^0 + \{Df_j(U, V)\}_{P+e_j, P} + \mathcal{M}_P(\{M^0\}_{\frac{2|P|+1-2d}{2}}; \{f, U, V\}_{P(d)}),$$

$$(5.23) \quad N_{j,P} = N_{j,P}^0 + \{Dg_j(U, V)\}_{P, P+e_j} + \mathcal{N}_P(\{N^0\}_{\frac{2|P|+1-2d}{2}}; \{g, U, V\}_{P(d)})$$

with

$$P(d) = 2|P| + 1 - d.$$

Here  $\mathcal{M}_P, \mathcal{N}_P$  are polynomials of which each has coefficients that depend only on  $P$ , and  $\{M^0\}_a$  stands for the set of coefficients  $M_Q^0$  with  $|Q| \leq a$  for a real number  $a \geq 0$ . Note that  $\mathcal{U}_{j,PQ} = \mathcal{V}_{j,QP} = 0$  when  $|P| + |Q| = 2$ , or  $\text{ord}(f, g) > |P| + |Q|$ . And  $\mathcal{M}_P = \mathcal{N}_P = 0$  when  $\text{ord}(f, g) > P(d)$ , by (5.13). We have

$$\{U_j(M(\xi\eta)\xi, N(\xi\eta)\eta)\}_{PQ} = \mu^{P-Q}U_{j,PQ} + \mathcal{U}_{j,PQ}(\{M, N\}_{\frac{|P|+|Q|-d}{2}}, \{U\}_{|P|+|Q|-2}).$$

Comparing coefficients in (5.14), (5.16), and using (5.20), we get for  $\ell = |P| + |Q|$

$$\begin{aligned}(\mu^{P-Q} - \mu_j)U_{j,PQ} &= \{f_j + Df_j(U, V)\}_{PQ} \\ &\quad + \mathcal{U}_{j,PQ}(\{M^0\}_{\frac{\ell+1-2d}{2}}, \{M, N\}_{\frac{\ell-d}{2}}; \{f, U, V\}_{\ell-2}).\end{aligned}$$

We have analogous formula for  $V_{j,QP}$ . Using (5.22), we obtain with  $|P| + |Q| = \ell$

$$(5.24) \quad (\mu^{P-Q} - \mu_j)U_{j,PQ} = \{f_j + Df_j(U, V)\}_{PQ} + \mathcal{U}_{j,PQ}(\{M^0, N^0\}_{\frac{\ell-d}{2}}; \{f, g, U, V\}_{\ell-2}),$$

$$(5.25) \quad (\mu^{Q-P} - \mu_j^{-1})V_{j,QP} = \{g_j + Dg_j(U, V)\}_{PQ} + \mathcal{V}_{j,QP}(\{M^0, N^0\}_{\frac{\ell-d}{2}}; \{f, g, U, V\}_{\ell-2}).$$

for  $\mu^{P-Q} \neq \mu_j$ , which are always solvable. Inductively, by using (5.24)-(5.25) and (5.22)-(5.23), we obtain unique solutions  $U, V, M, N$ . Moreover, the solutions and their dependence on the coefficients of  $f, g$  and small divisors have the form

$$(5.26) \quad (\mu^{P-Q} - \mu_j)U_{j,PQ} = \{f_j + Df_j(U, V)\}_{PQ} + \mathcal{U}_{j,PQ}^*(\delta_{\ell-2}, \{M^0, N^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2}),$$

$$(5.27) \quad (\mu^{Q-P} - \mu_j^{-1})V_{j,QP} = \{g_j + Dg_j(U, V)\}_{PQ} + \mathcal{V}_{j,QP}^*(\delta_{\ell-2}, \{M^0, N^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2}).$$

where  $\ell = |P| + |Q|$  and  $\mu^{P-Q} \neq \mu_j$ , and  $\delta_i$  is the union of  $\{\mu_1, \mu_1^{-1}, \dots, \mu_p, \mu_p^{-1}\}$  and

$$\left\{ \frac{1}{\mu^{A-B} - \mu_j} : |A| + |B| \leq i, j = 1, \dots, p, A, B \in \mathbf{N}^p \right\}.$$

This shows that for any  $M^0, N^0$  there exists a unique mapping  $\Psi$  transforms  $\sigma$  into  $\sigma^*$ . Furthermore,  $\mathcal{U}_{j,PQ}^*(t'; t''), \mathcal{V}_{j,QP}^*(t'; t'')$  are polynomials of which each has coefficients that depend only on  $j, P, Q$ , and they vanish at  $t'' = 0$ .

For later purpose, let us express  $M, N$  in terms of  $f, g$ . We substitute expressions (5.26)-(5.27) for  $U, V$  in (5.22)-(5.23) to obtain

$$(5.28) \quad M_{j,P} = M_{j,P}^0 + \{Df_j(U, V)\}_{P+e_j P} + \mathcal{M}_{j,P}^*(\delta_{P(d)}, \{M^0, N^0\}_{\frac{P(d)}{2}}; \{f, g\}_{P(d)}),$$

$$(5.29) \quad N_{j,P} = N_{j,P}^0 + \{Dg_j(U, V)\}_{PP+e_j} + \mathcal{N}_{j,P}^*(\delta_{P(d)}, \{M^0, N^0\}_{\frac{P(d)}{2}}; \{f, g\}_{P(d)}).$$

with  $f, g$  satisfying (5.13).

Assume that  $\tilde{\sigma} = \psi_0^{-1} \sigma^* \psi_0$  commutes with  $\hat{S}$ . By Corollary 4.6, we can decompose  $\psi_0 = HG^{-1}$  with  $G \in \mathcal{C}(\hat{S})$  and  $H \in \mathcal{C}^c(\hat{S})$ . Furthermore,  $G^{-1} \tilde{\sigma} G$  commutes with  $\hat{S}$  and  $H^{-1} \sigma^* H$ . By the uniqueness conclusion for the above  $\psi_0$ ,  $H$  must be the identity. This shows that  $\psi_0 \in \mathcal{C}(\hat{S})$ .

(i). Assume that we have normalized  $\sigma$ . We now use it to normalize the pair of involutions. Assume that  $\sigma = \tau_1 \tau_2$  and  $\tau_j^2 = I$ . Then  $\sigma^* = \tau_1^* \tau_2^*$ . Let  $T_0(\xi, \eta) := (\eta, \xi)$ . We have  $T_0(\sigma^*)^{-1} T_0 = T_0 \tau_1^* \sigma^* \tau_1^* T_0$ . By the above normalization,  $T_0(\sigma^*)^{-1} T_0$  commutes with  $\hat{S}$ . Therefore,  $\tau_1^* T_0$  belongs to the centralizer of  $\hat{S}$  and it must be of the form  $(\xi, \eta) \rightarrow (\xi \Lambda_1(\xi \eta), \eta \Lambda_1^*(\xi \eta))$ . Then  $(\tau_1^*)^2 = I$  implies that

$$\Lambda_1(\xi \eta (\Lambda_1 \Lambda_1^*)(\xi \eta)) \Lambda_1^*(\xi \eta) = 1.$$

The latter implies, by induction on  $d > 1$ , that  $\Lambda_1 \Lambda_1^* = 1 + O(d)$  for all  $d > 1$ , i.e.  $\Lambda_1 \Lambda_1^* = 1$ .

Let  $\tau_i^*$  be given by (5.4). We want to achieve  $\tilde{\Lambda}_{1j} \tilde{\Lambda}_{2j} = 1$  for  $\tilde{\tau}_i = \psi_0^{-1} \tau_i^* \psi_0$  by applying a transformation  $\psi_0$  in  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  that commutes with  $\hat{S}$ . According to Definition 4.3, it has the form

$$\psi_0: \xi_j = \tilde{\xi}_j(1 + a_j(\tilde{\zeta})), \quad \eta_j = \tilde{\eta}_j(1 - a_j(\tilde{\zeta}))$$

with  $a_j(0) = 0$ . Here  $\tilde{\zeta}_j := \tilde{\xi}_j \tilde{\eta}_j$  and  $\tilde{\zeta} := (\tilde{\zeta}_1, \dots, \tilde{\zeta}_p)$ . Computing the products  $\zeta$  in  $\tilde{\zeta}$  and solving  $\tilde{\zeta}$  in  $\zeta$ , we obtain

$$\psi_0^{-1}: \tilde{\xi}_j = \xi_j(1 + b_j(\zeta))^{-1}, \quad \tilde{\eta}_j = \eta_j(1 - b_j(\zeta))^{-1}.$$

Note that  $(a_j^2)_P = \mathcal{A}_{j,P}(\{a\}_{|P|-1})$ , and

$$\xi_j \eta_j = \tilde{\xi}_j \tilde{\eta}_j(1 - a_j^2(\tilde{\zeta})), \quad \tilde{\xi}_j \tilde{\eta}_j = \xi_j \eta_j(1 - b_j^2(\zeta))^{-1}.$$

From  $\psi_0^{-1} \psi_0 = I$ , we get

$$(5.30) \quad b_j(\zeta) = a_j(\tilde{\zeta}), \quad b_{j,P} = a_{j,P} + \mathcal{B}_{j,P}(\{a\}_{|P|-1}).$$

By a simple computation we see that  $\tilde{\tau}_i = \psi_0^{-1} \tau_i^* \psi_0$  is given by

$$\tilde{\xi}'_j = \tilde{\eta}_j \tilde{\Lambda}_{ij}(\tilde{\zeta}), \quad \tilde{\eta}'_j = \tilde{\xi}_j \tilde{\Lambda}_{ij}^{-1}(\tilde{\zeta})$$

with

$$\tilde{\Lambda}_{1j} \tilde{\Lambda}_{2j}(\tilde{\zeta}) = (\Lambda_{1j} \Lambda_{2j})(\zeta)(1 + b_j(\zeta'))^{-2}(1 - a_j(\tilde{\zeta}))^2.$$

Here  $\zeta'_j = \zeta_j(1 - a_j^2(\tilde{\zeta}))$ . Using (5.30) and the implicit function theorem, we determine  $a_j$  uniquely to achieve  $\tilde{\Lambda}_{1j} \tilde{\Lambda}_{2j} = 1$ .

To identify the transformations that preserve the form of  $\tilde{\tau}_1, \tilde{\tau}_2$ , we first verify that each element  $\psi_1 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$  preserves that form. According to (4.4), we have

$$\begin{aligned}\psi_1: \xi_j &= \tilde{\xi}_j \tilde{a}_j(\tilde{\zeta}), & \eta_j &= \tilde{\eta}_j \tilde{a}_j(\tilde{\zeta}), \\ \psi_1^{-1}: \tilde{\xi}_j &= \xi_j \tilde{b}_j(\zeta), & \tilde{\eta}_j &= \eta_j \tilde{b}_j(\zeta), \\ & \tilde{b}_j(\zeta) \tilde{a}_j(\tilde{\zeta}) &= 1.\end{aligned}$$

This shows that  $\psi_1^{-1} \tilde{\tau}_i$  is given by

$$\tilde{\xi}'_j = \tilde{\Lambda}_{ij}(\zeta) \tilde{b}_j(\zeta) \eta_j, \quad \tilde{\eta}'_j = \tilde{\Lambda}_{ij}^{-1}(\zeta) \tilde{b}_j(\zeta) \xi_j.$$

Then  $\psi_1^{-1} \tilde{\tau}_i \psi_1$  is given by

$$\tilde{\xi}'_j = \tilde{\Lambda}_{ij}(\zeta) \tilde{\eta}_j, \quad \tilde{\eta}'_j = \tilde{\Lambda}_{ij}^{-1}(\zeta) \tilde{\xi}_j.$$

Since  $\zeta_j = \tilde{\zeta}_j \tilde{a}_j^2(\tilde{\zeta})$ , then  $\psi_1^{-1} \tilde{\tau}_i \psi_1$  still satisfy (5.5). Conversely, suppose that  $\psi_1$  preserves the forms of  $\tilde{\tau}_1, \tilde{\tau}_2$ . We apply Corollary 4.6 to decompose  $\psi_1 = \phi_1 \phi_0^{-1}$  with  $\phi_0 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$  and  $\phi_1 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ . Since we just proved that each element in  $\mathcal{C}(\hat{T}_1, \hat{T}_2)$  preserves the form of  $\tilde{\tau}_i$ , then  $\phi_1 = \psi_1 \phi_0$  also preserves the forms of  $\tilde{\tau}_1, \tilde{\tau}_2$ . On the other hand, we have shown that there exists a unique mapping in  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  which transforms  $\{\tau_1^*, \tau_2^*\}$  into  $\{\tilde{\tau}_1, \tilde{\tau}_2\}$ . This shows that  $\phi_0 = I$ . We have verified all assertions in (i).

(ii). It is easy to see that  $\mathcal{C}^c(\hat{S})$  and  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  are invariant under conjugacy by  $\rho$ . We have  $\Psi^{-1} \sigma \Psi = \sigma^*$  and  $\Psi \in \mathcal{C}^c(\hat{S})$ . Note that  $\rho \sigma \rho = \sigma^{-1}$  and  $\rho \sigma^* \rho$  have the same form as of  $(\sigma^*)^{-1}$ , i.e. they are in  $\mathcal{C}^c(\hat{S})$  and have the same linear part. We have  $\rho \Psi \rho \sigma \rho \Psi^{-1} \rho = \rho (\sigma^*)^{-1} \rho$ . The uniqueness of  $\Psi$  implies that  $\rho \Psi \rho = \Psi$  and  $\tau_2^* = \rho \tau_1^* \rho$ . Thus, we obtain relations (5.7)-(5.10). Analogously,  $\rho \psi_0 \rho$  is still in  $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ , and  $\rho \phi_0 \rho$  preserves the form of  $\tilde{\tau}_1, \tilde{\tau}_2$ . Thus  $\rho \psi_0 \rho = \psi_0$  and  $\tilde{\tau}_2 = \rho \tilde{\tau}_1 \rho$ , which gives us (5.11).  $\square$

We will also need the following uniqueness result.

**Corollary 5.3.** *Suppose that  $\sigma$  has a non-resonant linear part  $\hat{S}$ . Let  $\Psi$  be the unique formal mapping in  $\mathcal{C}^c(\hat{S})$  such that  $\Psi^{-1} \sigma \Psi \in \mathcal{C}(\hat{S})$ . If  $\tilde{\Psi} \in \mathcal{C}^c(\hat{S})$  is a polynomial map of degree at most  $d$  such that  $\tilde{\Psi}^{-1} \sigma \tilde{\Psi}(\xi, \eta) = \tilde{\sigma}(\xi, \eta) + O(|(\xi, \eta)|^{d+1})$  and  $\tilde{\sigma} \in \mathcal{C}(\hat{S})$ , then  $\tilde{\Psi}$  is unique. In fact,  $\Psi - \tilde{\Psi} = O(d+1)$ .*

*Proof.* The proof is contained in the proof of Proposition 5.2. Let us recap it by using (5.26)-(5.27) and the proposition. We take a unique normalized mapping  $\Phi$  such that  $\Phi^{-1} \tilde{\Psi}^{-1} \sigma \tilde{\Psi} \Phi \in \mathcal{C}(\hat{S})$ . By (5.26)-(5.27),  $\Phi = I + O(d+1)$ . From Proposition 5.2 it follows that  $\psi_0 := \tilde{\Psi} \Phi \Psi^{-1} \in \mathcal{C}(\hat{S})$ . We obtain  $\tilde{\Psi} \Phi = \psi_0 \Psi$ . Thus  $\psi_0 \Psi = \tilde{\Psi} + O(d+1)$ . Since  $\psi_0 \in \mathcal{C}(\hat{S})$ , and  $\Psi, \tilde{\Psi}$  are in  $\mathcal{C}^c(\hat{S})$ , we conclude that  $\Psi = \tilde{\Psi} + O(d+1)$ .  $\square$

For clarity, we state the following uniqueness results on normalization.

**Corollary 5.4.** *Let  $\sigma$  have a non-resonant linear part and let  $\sigma$  be given by*

$$\xi'_j = M_j^0(\xi \eta) \xi_j + f_j^0(\xi, \eta), \quad \eta'_j = N_j^0(\xi \eta) \eta_j + g_j^0(\xi, \eta).$$

*Let  $\Psi = I + (U, V) \in \mathcal{C}^c(\hat{S})$  and let  $\sigma^* = \Psi^{-1} \sigma \Psi$  be given by*

$$\xi'_j = M_j(\xi \eta) \xi_j + f_j(\xi, \eta), \quad \eta'_j = N_j(\xi \eta) \eta_j + g_j(\xi, \eta).$$

Suppose that  $(f^0, g^0)$  and  $(f, g)$  are in  $\mathcal{C}_2^c(\hat{S})$ ,  $\text{ord}(f^0, g^0) \geq d$ ,  $\text{ord}(f_j, g_j) \geq d$ , and  $d \geq 2$ . Then  $\text{ord}(U, V) \geq d$  and

$$(5.31) \quad M_{j,P} = M_{j,P}^0, \quad N_{j,P} = N_{j,P}^0, \quad 1 \leq 2|P| + 1 < 2d - 1.$$

*Proof.* By Corollary 5.3, we know that  $\text{ord}(U, V) \geq d$ . Expanding both sides of  $\sigma\Psi = \Psi\sigma^*$  for terms of degree less than  $2d - 1$ , we obtain

$$\begin{aligned} M_j^0(\xi\eta)(\xi_j + U_j(\xi, \eta)) + DM_j^0(\xi\eta)(\xi V + \eta U)\xi_j + f_j^0(\xi, \eta) \\ = M_j(\xi\eta)\xi_j + f_j(\xi, \eta) + U_j(M(\xi\eta)\xi, N(\xi\eta)\eta) + O(2d - 1). \end{aligned}$$

Note that  $\xi_i V_i(\xi, \eta)\xi_j$  and  $\eta_i U_i(\xi, \eta)\xi_j$  and  $U_j(M(\xi\eta)\xi, N(\xi\eta)\eta)$  do not contain terms of the form  $\xi^Q \eta^Q \xi_j$ . Comparing the coefficients of  $\xi^P \eta^P \xi_j$  for  $2|P| + 1 < 2d - 1$ , we obtain the first identity in (5.31). The second identity can be obtained similarly.  $\square$

When  $p = 1$ , Proposition 5.2 is due to Moser and Webster [MW83]. In fact, they achieved

$$\tilde{M}_1(\zeta_1) = e^{\delta(\xi_1 \eta_1)^s}.$$

Here  $\delta = 0, \pm 1$  for the elliptic case and  $\delta = 0, \pm i$  for the hyperbolic case when  $\mu_1$  is not a root of unity, i.e.  $\gamma$  is *non-exceptional*. In particular the normal form is always convergent, although the normalizing transformations are generally divergent for the hyperbolic case.

Let us find out further normalization that can be performed to preserve the form of  $\sigma^*$ . In Proposition 5.2, we have proved that if  $\sigma$  is tangent to  $\hat{S}$ , there exists a unique  $\Psi \in \mathcal{C}^c(\hat{S})$  such that  $\Psi^{-1}\sigma\Psi$  is an element  $\sigma^*$  in the centralizer of  $\hat{S}$ . Suppose now that  $\sigma = \tau_1\tau_2$  while  $\tau_i$  is tangent to  $\hat{T}_i$ . Let  $\tau_i^* = \Psi^{-1}\tau_i\Psi$ . We have also proved that there is a unique  $\psi_0 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$  such that  $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$ ,  $i = 1, 2$ , are of the form (5.5), i.e.

$$\begin{aligned} \tilde{\tau}_i: \xi_j' &= \tilde{\Lambda}_{ij}(\zeta)\eta_j, & \eta_j' &= \tilde{\Lambda}_{ij}^{-1}(\zeta)\xi_j; \\ \tilde{\sigma}: \xi_j' &= \tilde{M}_j(\zeta)\xi_j, & \eta_j' &= \tilde{M}_j^{-1}(\zeta)\eta_j. \end{aligned}$$

Here  $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$ ,  $\tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}$  and  $\tilde{M}_j = \tilde{\Lambda}_{1j}^2$ . We still have freedom to further normalize  $\tilde{\tau}_1, \tilde{\tau}_2$  and to preserve their forms. However, any new coordinate transformation must be in  $\mathcal{C}(\hat{T}_1, \hat{T}_2)$ , i.e. it must have the form

$$\psi_1: \xi_j \rightarrow a_j(\xi\eta)\xi_j, \quad \eta_j \rightarrow a_j(\xi\eta)\eta_j.$$

When  $\tau_{2j} = \rho\tau_{1j}\rho$ , we require that  $\psi_1$  commutes with  $\rho$ , i.e.

$$a_e = \bar{a}_e, \quad a_h = \bar{a}_h, \quad a_s = \bar{a}_{s+s_*}.$$

In  $\zeta$  coordinates, the transformation  $\psi_1$  has the form

$$(5.32) \quad \varphi: \zeta_j \rightarrow b_j(\zeta)\zeta_j, \quad 1 \leq j \leq p$$

with  $b_j = a_j^2$ . Therefore, the mapping  $\varphi$  needs to satisfy

$$b_e > 0, \quad b_h > 0, \quad b_s = \bar{b}_{s+s_*}.$$

Recall from (5.7)-(5.10) the reality conditions on  $\tilde{M}_j$

$$\begin{aligned}\overline{\tilde{M}_e \circ \rho_z} &= \tilde{M}_e, & 1 \leq e \leq e_*; \\ \overline{\tilde{M}_h \circ \rho_z} &= \tilde{M}_h^{-1}, & e_* < h \leq h_* + e_*; \\ \overline{\tilde{M}_{s_*+s}} &= \tilde{M}_s^{-1} \circ \rho_z, & h_* + e_* < s \leq p - s_*.\end{aligned}$$

Here

$$\rho_z: \zeta_j \rightarrow \bar{\zeta}_j, \quad \zeta_s \rightarrow \bar{\zeta}_{s+s_*}, \quad \zeta_{s+s_*} \rightarrow \bar{\zeta}_s$$

for  $1 \leq j \leq e_* + h_*$  and  $e_* + h_* < s \leq p - s_*$ .

Therefore, our normal form problem leads to another normal form problem which is interesting in its own right. To formulate a new normalization problem, let us define

$$(5.33) \quad (\log \tilde{M})_j(\zeta) := \begin{cases} \log(\tilde{M}_j(\zeta)/\tilde{M}_j(0)), & 1 \leq j \leq e_*, \\ -i \log(\tilde{M}_j(\zeta)/\tilde{M}_j(0)), & e_* < j \leq p. \end{cases}$$

Let  $F = \log \tilde{M} := ((\log \tilde{M})_1, \dots, (\log \tilde{M})_p)$ . Then the reality conditions on  $\tilde{M}$  become

$$(5.34) \quad F = \rho_z F \rho_z.$$

The transformations (5.32) will then satisfy

$$\rho_z \varphi \rho_z = \varphi, \quad b_j(0) > 0, \quad 1 \leq j \leq e_* + h_*.$$

By using  $\log \tilde{M}$ , we have transformed the reality condition on  $M$  into a linear condition (5.34). This will be useful to further normalize  $\tilde{M}$ . Therefore, when  $F'(0)$  is furthermore diagonal and invertible and its  $j$ th diagonal entry is positive for  $j = e, h$ , we apply a dilation  $\varphi$  satisfying the above condition so that  $F$  is tangent to the identity. Then any further change of coordinates must be tangent to the identity too. Thus, we need to normalize the formal holomorphic mapping  $F$  by composition  $F \circ \varphi$ , for which we study in next subsection.

**5.2. A normal form for maps tangent to the identity.** Let us consider a germ of holomorphic mapping  $F(\zeta)$  in  $\mathbf{C}^p$  with an invertible linear part  $\mathbf{A}\zeta$  at the origin. According to the inverse function theorem, there exists a holomorphic mapping  $\Psi$  with  $\Psi(0) = 0$ ,  $\Psi'(0) = I$  such that  $F \circ \Psi(\zeta) = \mathbf{A}\zeta$ . On the other hand, if we impose some restrictions on  $\Psi$ , we can no longer linearize  $F$  in general.

To focus on applications to CR singularity and to limit the scope of our investigation, we now deliberately restrict our analysis to the simplest case :  $F$  is tangent to the identity. We shall apply our result to  $F = \log \tilde{M}$  as defined in the previous subsection. In what follows, we shall devise a normal form of such an  $F$  under right composition by  $\Psi$  that preserve all coordinate hyperplanes, i.e.  $\Psi_j(\zeta) = \zeta_j \tilde{\Psi}_j(\zeta)$ ,  $j = 1, \dots, p$ .

**Lemma 5.5.** *Let  $F$  be a formal holomorphic map of  $\mathbf{C}^p$  that is tangent to the identity at the origin.*

- (i) *There exists a unique formal biholomorphic map  $\psi$  which preserves all  $\zeta_j = 0$  such that  $\hat{F} := F \circ \psi$  has the form*

$$(5.35) \quad \hat{F}(\zeta) = \zeta + \hat{f}(\zeta), \quad \hat{f}(\zeta) = O(|\zeta|^2); \quad \partial_{\zeta_j} \hat{f}_j = 0, \quad 1 \leq j \leq p.$$

(ii) If  $F$  is convergent, the  $\psi$  in (i) is convergent. If  $F$  commutes with  $\rho_z$ , so does the  $\psi$ .

(iii) The formal normal form in (i) has the form

$$(5.36) \quad \hat{f}_{j,Q} = f_{j,Q} - \{Df_j \cdot f\}_Q + \mathcal{F}_{j,Q}(\{f\}_{|Q|-2}), \quad q_j = 0, \quad |Q| > 1.$$

Here  $\mathcal{F}_{j,Q}$  are universal polynomials and vanish at 0.

*Proof.* (i) Write  $F(\zeta) = \zeta + f(\zeta)$  and

$$\psi: \zeta'_j = \zeta_j + \zeta_j g_j(\zeta), \quad g_j(0) = 0.$$

For  $\hat{F} = F \circ \psi$ , we need to solve for  $\hat{f}, g$  from

$$\hat{f}_j(\zeta) = \zeta_j g_j(\zeta) + f_j \circ \psi(\zeta).$$

Fix  $Q = (q_1, \dots, q_p) \in \mathbf{N}^p$  with  $|Q| > 1$ . We obtain unique solutions

$$(5.37) \quad g_{j,Q-e_j} = -\{f_j(\psi(\zeta))\}_Q, \quad q_j > 0,$$

$$(5.38) \quad \hat{f}_{j,Q} := \{f_j(\psi(\zeta))\}_Q, \quad q_j = 0.$$

We first obtain  $g_{j,Q-e_j} = -f_{j,Q} + \mathcal{G}_Q(\{f\}_{|Q|-1}, \{g\}_{|Q|-2})$ . This determines

$$(5.39) \quad g_{j,Q-e_j} = -f_{j,Q} + \mathcal{G}_Q(\{f\}_{|Q|-1}).$$

Next, we expand  $f_j(\psi(\zeta)) = f_j(\zeta) + Df_j(\zeta) \cdot (\zeta_1 g_1(\zeta), \dots, \zeta_p g_p(\zeta)) + \mathcal{R}_2 f_j(\zeta; \zeta g(\zeta))$ . The last term, with  $\text{ord } g \geq 1$ , has the form

$$\{\mathcal{R}_2 f_j(\zeta; \zeta g(\zeta))\}_Q = \mathcal{F}_{j,Q}(\{f\}_{|Q|-2}, \{g\}_{|Q|-2}) = \tilde{\mathcal{F}}_{j,Q}(\{f\}_{|Q|-2}).$$

Combining (5.38), the expansion, and (5.39), we obtain (5.36).

(ii) Assume that  $F$  is convergent. Define  $\bar{h}(\zeta) = \sum |h_Q| \zeta^Q$ . We obtain for every multi-index  $Q = (q_1, \dots, q_p)$  and for every  $j$  satisfying  $q_j \geq 1$

$$\bar{g}_{j,Q-e_j} \leq \{\bar{f}_j(\zeta_1 + \zeta_1 \bar{g}_1(\zeta), \dots, \zeta_p + \zeta_p \bar{g}_p(\zeta))\}_Q.$$

Set  $w(\zeta) = \sum \zeta_k \bar{g}_k(\zeta)$ . We obtain

$$w(\zeta) \prec \sum \bar{f}_j(\zeta_1 + w(\zeta), \dots, \zeta_p + w(\zeta)).$$

Note that  $f_j(\zeta) = O(|\zeta|^2)$  and  $w(0) = 0$ . By the Cauchy majorization and the implicit function theorem,  $w$  and hence  $g, \psi, \hat{f}$  are convergent.

(iii) Assume that  $\rho_z F \rho_z = F$ . Then  $\rho_z \hat{F} \rho_z$  is normalized,  $\rho_z \psi \rho_z$  is tangent to the identity, and the  $j$ th component of  $\rho_z \hat{F} \rho_z(\zeta) - \zeta$  is independent of  $\zeta_j$ . Thus  $\rho_z \psi \rho_z$  normalizes  $F$  too. By the uniqueness of  $\psi$ , we obtain  $\rho_z \psi \rho_z = \psi$ .

By rewriting (5.38), we obtain

$$(5.40) \quad \hat{f}_{j,Q} = f_{j,Q} + \{f_j(\psi) - f_j\}_Q = f_{j,Q} + \mathcal{F}'_{j,Q}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}).$$

From (5.37), it follows that

$$g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}_{k,Q-e_k}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}), \quad |Q| > 1.$$

Note that  $\{g\}_0 = 0$  and  $\{f\}_1 = 0$ . Using the identity repeatedly, we obtain  $g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}_{k,Q-e_k}^*(\{f\}_{|Q|-1})$ . Therefore, we can rewrite (5.40) as (5.36).  $\square$

**5.3. A unique formal normal form of a reversible map  $\sigma$ .** We now state a normal form for  $\{\tau_1, \tau_2, \rho\}$  under a condition on the third-order invariants of  $\sigma$ .

**Theorem 5.6.** *Let  $\tau_1, \tau_2$  be a pair of holomorphic involutions with linear parts  $\hat{T}_i$ . Let  $\sigma = \tau_1\tau_2$ . Assume that the linear part of  $\sigma$  is*

$$\hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p$$

and  $\mu_1, \dots, \mu_p$  are non-resonant. Let  $\Psi \in \mathcal{C}^c(\hat{S})$  be the unique formal mapping such that

$$\begin{aligned} \tau_i^* &= \Psi^{-1} \tau_i \Psi: \xi'_j = \Lambda_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \Lambda_{ij}(\xi\eta)^{-1} \xi_j; \\ \sigma^* &= \Psi^{-1} \sigma \Psi: \xi'_j = M_j(\xi\eta) \xi_j, \quad \eta'_j = M_j(\xi\eta)^{-1} \eta_j \end{aligned}$$

with  $M_j = \Lambda_{1j} \Lambda_{2j}^{-1}$ . Suppose that  $\sigma$  satisfies the condition that  $\log M$  is tangent to the identity.

(i) Then there exists an invertible formal map  $\psi_1 \in \mathcal{C}(\hat{S})$  such that

$$(5.41) \quad \begin{aligned} \hat{\tau}_i &= \psi_1^{-1} \tau_i^* \psi_1: \xi'_j = \hat{\Lambda}_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \hat{\Lambda}_{ij}(\xi\eta)^{-1} \xi_j; \\ \hat{\sigma} &= \psi_1^{-1} \sigma^* \psi_1: \xi'_j = \hat{M}_j(\xi\eta) \xi_j, \quad \eta'_j = \hat{M}_j(\xi\eta)^{-1} \eta_j. \end{aligned}$$

Here  $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$ , and  $\hat{T}_i$  is the linear part of  $\hat{\tau}_i$ . Moreover,  $\log \hat{M}_j(\zeta) - \zeta_j = O(2)$  is independent of  $\zeta_j$  for each  $j$ .

(ii) The centralizer of  $\{\hat{\tau}_1, \hat{\tau}_2\}$  consists of  $2^p$  dilations  $(\xi, \eta) \rightarrow (a\xi, a\eta)$  with  $a_j = \pm 1$ . And  $\hat{\Lambda}_{ij}$  are unique. If  $\Lambda_{ij}$  are convergent, then  $\psi_1$  is convergent too.

(iii) Suppose that  $\hat{\sigma}$  is divergent. If  $\sigma$  is formally equivalent to a mapping  $\tilde{\sigma} \in \mathcal{C}(\hat{S})$  then  $\tilde{\sigma}$  must be divergent too.

(iv) Let  $\rho$  be given by (3.8) and let  $\tau_2 = \rho\tau_1\rho$ . Then the above  $\Psi$  and  $\psi_1$  commute with  $\rho$ . Moreover,  $\hat{\tau}_i, \hat{\sigma}$  are unique.

*Proof.* Assertions in (i) are direct consequences of Proposition 5.2 and Lemma 5.5 in which  $F$  is the  $\tilde{M}$  in Proposition 5.2. The assertion in (ii) on the centralizer of  $\{\hat{\tau}_1, \hat{\tau}_2\}$  is obtained from (5.6) of Proposition 5.2 in which  $\tilde{\Lambda}_{ij} = \hat{\Lambda}_{ij}$ . Indeed, by (5.6), if  $\psi$  preserves  $\{\hat{\tau}_1, \hat{\tau}_2\}$ , then  $\psi(\xi, \eta) = (c(\xi\eta)\xi, c(\xi\eta)\eta)$  and  $\hat{M}_j(c^2(\xi\eta)\xi\eta) = \hat{M}_j(\xi\eta)$ . This shows that  $\hat{M} \circ \tilde{\psi} = \hat{M}$  for  $\tilde{\psi}(\zeta) = c^2(\zeta)\zeta$ . Since  $\hat{M} - \hat{M}(0)$  is invertible then  $\tilde{\psi}$  is the identity, i.e.  $c_j = \pm 1$ . Now (iii) follows from (ii) too. Indeed, suppose  $\sigma$  is formally equivalent to some convergent

$$\tilde{\sigma}: \xi_j = \tilde{M}_j(\xi\eta) \xi_j, \quad \eta'_j = \tilde{M}_j(\xi\eta)^{-1} \eta_j.$$

Then by the assumption on the linear part of  $\log M$ , we can apply a dilation to achieve that  $(\log \tilde{M})'(0)$  is tangent to the identity. By Lemma 5.5, there exists a unique convergent mapping  $\varphi: \zeta'_j = b_j(\zeta) \zeta_j$  ( $1 \leq j \leq p$ ) with  $b_j(0) = 1$  such that  $\log \tilde{M} \circ \varphi$  is in the normal form  $\log M_*$ . Then

$$(\xi'_j, \eta'_j) = (b_j^{1/2}(\xi\eta) \xi_j, b_j^{1/2}(\xi\eta) \eta_j), \quad 1 \leq j \leq p$$

transforms  $\tilde{\sigma}$  into a convergent mapping  $\sigma_*$ . Since the normal form for  $\log M$  is unique, then  $\hat{\sigma} = \sigma_*$ . In particular,  $\hat{\sigma}$  is convergent.

(iv). Note that  $\rho\sigma\rho = \sigma^{-1}$ . Also  $\rho(\sigma^*)^{-1}\rho$  has the same form as  $\sigma^*$ . By  $(\rho\Psi^{-1}\rho)\sigma(\rho\Psi\rho) = (\rho\sigma^*\rho)^{-1}$ , we conclude that  $\rho\psi_1\rho = \Psi$ . The rest of assertions can be verified easily.  $\square$

Note that  $M^{-1}(\zeta)$  is also normalized in the sense that  $\log M_j^{-1}(\zeta) + \zeta_j = O(|\zeta|^2)$  is independent of  $\zeta_j$ . Under the condition that  $\log M$  is tangent to the identity, the above theorem completely settles the formal classification of  $\{\tau_1, \tau_2, \rho\}$ . It also says that **the normal form  $\hat{\tau}_1, \hat{\tau}_2$  can be achieved by a convergent transformation, if and only if  $\sigma^*$  can be achieved by some convergent transformation**, i.e. the  $\Psi$  in the theorem is convergent.

However, we would like state clear that our results do not rule out the case where a refined normal form for  $\{\tau_1^*, \tau_2^*, \rho\}$  is achieved by convergent transformation, while  $\Psi$  is divergent, when  $\log M$  is tangent to the identity.

**5.4. An algebraic manifold with linear  $\sigma$ .** We conclude the section showing that when  $\tau_1, \tau_2$  are normalized as in this section,  $\{\tau_{ij}\}$  might still be very general; in particular  $\{\tau_{1j}, \rho\}$  cannot always be simultaneously linearized even at the formal level. This is one of main differences between  $p = 1$  and  $p > 1$ .

**Example 5.7.** Let  $p = 2$ . Let  $\phi$  be a holomorphic mapping of the form

$$\phi: \xi'_i = \xi_i + q_i(\xi, \eta), \quad \eta'_i = \eta_i + \lambda_i^{-1} q_i(T_1(\xi, \eta)), \quad i = 1, 2.$$

Here  $q_i$  is a homogeneous quadratic polynomial map and

$$T_1(\xi, \eta) = (\lambda_1 \eta_1, \lambda_2 \eta_2, \lambda_1^{-1} \xi_1, \lambda_2^{-1} \xi_2).$$

Let  $\tau_{1j} = \phi T_{1j} \phi^{-1}$  and  $\tau_{2j} = \rho \tau_{1j} \rho$ . Then  $\phi$  commutes with  $T_1$  and  $\tau_1 = T_1$ . In particular  $\tau_2 = \rho T_1 \rho$  and  $\sigma = \tau_1 \tau_2$  are in linear normal forms. However,  $\tau_{11}$  is given by

$$\begin{aligned} \xi'_1 &= \lambda_1 \eta_1 - q_1(\lambda \eta, \lambda^{-1} \xi) + q_1(\lambda_1 \eta_1, \xi_2, \lambda_1^{-1} \xi_1, \eta_2) + O(3), \\ \xi'_2 &= \xi_2 - q_2(\xi, \eta) + q_2(\lambda_1 \eta_1, \xi_2, \lambda_1^{-1} \xi_1, \eta_2) + O(3), \\ \eta'_1 &= \lambda_1^{-1} \xi_1 - \lambda_1^{-1} q_1(\xi, \eta) + \lambda_1^{-1} q_1(\xi_1, \lambda_2 \eta_2, \eta_1, \lambda_2^{-1} \xi_2) + O(3), \\ \eta'_2 &= \eta_2 - \lambda_2^{-1} q_2(\lambda \eta, \lambda^{-1} \xi) + \lambda_2^{-1} q_2(\xi_1, \lambda_2 \eta_2, \eta_1, \lambda_2^{-1} \xi_2) + O(3). \end{aligned}$$

Notice that the common zero set  $V$  of  $\xi_1 \eta_1$  and  $\xi_2 \eta_2$  is invariant under  $\tau_1, \tau_2, \sigma$  and  $\rho$ . In fact, they are linear on  $V$ . However, for  $(\xi', \eta') = \tau_{11}(\xi, \eta)$ , we have

$$\begin{aligned} \xi'_1 \eta'_1 &= -\eta_1 q_1(0, \xi_2, \eta) + \eta_1 q_1(0, \lambda_2 \eta_2, \eta_1, \lambda_2^{-1} \xi_2) - \lambda_1^{-1} \xi_1 q_1(0, \lambda_2 \eta_2, \lambda^{-1} \xi) \\ &\quad + \lambda_1^{-1} \xi_1 q_1(0, \xi_2, \lambda_1^{-1} \xi_1, \eta_2) \mod (\xi_1 \eta_1, \xi_2 \eta_2, O(4)). \end{aligned}$$

For a generic  $q$ ,  $\tau_{11}$  does not preserve  $V$ .

By Proposition 5.2, when the above linear  $\sigma$  is non-resonant,  $\{\tau_{11}, \tau_{12}, \rho\}$  is not linearizable. By a simple computation, we can verify that  $\sigma_j = \tau_{1j} \tau_{2j}$  for  $j = 1, 2$  do not commute with each other. In fact, we proved in [GS15] that if the  $\mu_1, \dots, \mu_p$  are nonresonant,  $\sigma_j$  commute pairwise, and  $\sigma$  is linear as above, then  $\tau_{1j}$  must be linear.

## 6. DIVERGENCE OF ALL NORMAL FORMS OF A REVERSIBLE MAP $\sigma$

Unlike the Birkhoff normal form for a Hamiltonian system, the Poincaré-Dulac normal form is not unique for a general  $\sigma$ ; it just belongs to the centralizer of the linear part  $S$  of  $\sigma$ . One can obtain a divergent normal form easily from any non-linear Poincaré-Dulac normal form of  $\sigma = \tau_1 \tau_2$  by conjugating with a divergent transformation in the centralizer of  $S$ ; see (5.6). We have seen how the small divisors enter in the computation of the normalizing



transformations via (5.26)-(5.27) and (5.22)-(5.23) in the computation of the normal forms. To see the effect of small divisors on normal forms, we first assume a condition, to be achieved later, on the third order invariants of  $\sigma$  and then we shall need to modify the normalization procedure. We will use two sequences of normalizing mappings to normalize  $\sigma$ . The composition of normalized mappings might not be normalized. Therefore, the new normal form  $\tilde{\sigma}$  might not be the  $\sigma^*$  in Proposition 5.2. We will show that this  $\tilde{\sigma}$ , after it is transformed into the normal form  $\hat{\sigma}$  in Theorem 5.6 (i), is divergent. Using the divergence of  $\hat{\sigma}$ , we will then show that any other normal forms of  $\sigma$  that are in the centralizer of  $S$  must be divergent too. This last step requires a convergent solution given by Lemma 5.5.

Our goal is to see a small divisor in a normal form  $\tilde{\sigma}$ ; however they appear as a product. This is more complicated than the situation for the normalizing transformations, where a small divisor appears in a much simple way. In essence, a small divisor problem occurs naturally when one applies a Newton iteration scheme for a convergence proof. For a small divisor to show up in the normal form, we have to go beyond the Newton iteration scheme, measured in the degree or order of approximation in power series. Therefore, we first refine the formulae (5.22).

**Lemma 6.1.** *Let  $\sigma$  be a holomorphic mapping, given by*

$$\xi'_j = M_j^0(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = N_j^0(\xi\eta)\eta_j + g_j(\xi, \eta), \quad 1 \leq j \leq p.$$

Here  $M_j^0(0) = \mu_j = N_j^0(0)^{-1}$ . Suppose that  $\text{ord}(f, g) \geq d \geq 4$  and  $I + (f, g) \in \mathcal{C}^c(S)$ . There exist unique polynomials  $U, V$  of degree at most  $2d - 1$  such that  $\Psi = I + (U, V) \in \mathcal{C}^c(S)$  transforms  $\sigma$  into

$$\sigma^*: \xi' = M(\xi\eta)\xi + \tilde{f}(\xi, \eta), \quad \eta' = N(\xi\eta)\eta + \tilde{g}(\xi, \eta)$$

with  $I + (\tilde{f}, \tilde{g}) \in \mathcal{C}^c(S)$  and  $\text{ord}(\tilde{f}, \tilde{g}) \geq 2d$ . Moreover,

$$(6.1) \quad U_{j,PQ} = (\mu^{P-Q} - \mu_j)^{-1} \left\{ f_{j,PQ} + \mathcal{U}_{j,PQ}^*(\delta_{\ell-2}, \{M^0, N^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2}) \right\},$$

$$(6.2) \quad V_{j,QP} = (\mu^{Q-P} - \mu_j^{-1})^{-1} \left\{ g_{j,QP} + \mathcal{V}_{j,QP}^*(\delta_{\ell-2}, \{M^0, N^0\}_d; \{f, g\}_{\ell-2}) \right\},$$

for  $2 \leq |P| + |Q| = \ell \leq 2d - 1$  and  $\mu^{P-Q} \neq \mu_j$ . In particular,  $\text{ord}(U, V) \geq d$ . Also,

$$(6.3) \quad M_{j,P} = M_{j,P}^0, \quad 2|P| + 1 < 2d - 1,$$

$$(6.4) \quad M_{j,P} = M_{j,P}^0 + \{Df_j(U, V)\}_{(P+e_j)P}, \quad 2|P| + 1 = 2d - 1.$$

Assume further that

$$(6.5) \quad (M^0)'(0) = \text{diag}(\mu_1, \dots, \mu_p).$$

Then for  $2|P| + 1 = 2d + 1$ , we have

$$(6.6) \quad M_{j,P} = M_{j,P}^0 + \mu_j \left\{ 2(U_j V_j)_{PP} + (U_j^2)_{(P+e_j)(P-e_j)} \right\} + \{Df_j(U, V)\}_{(P+e_j)P}.$$

**Remark 6.2.** Note that (6.3) follows from (5.31). Formulae (6.3), (6.4), (6.6) give us an effective way to compute the Poincaré-Dulac normal form. Although (6.4) contains small divisors, it will be more convenient to associate small divisors to (6.6) when we have 3 elliptic components in  $\sigma$ .

*Proof.* Let  $D_i$  denote  $\partial_{\zeta_i}$ . Let  $Du(\xi, \eta)$  and  $Dv(\zeta)$  denote the gradients of two functions. Let us expand both sides of the  $\xi_j$  components of  $\Psi\sigma^* = \sigma\Psi$  for terms of degree  $2d+2$ . For its left-hand side, Corollary 5.3 implies that  $\text{ord}(U, V) \geq d$  and we can use  $\text{ord } DU_j \cdot (\tilde{f}, \tilde{g}) \geq 2d-1+d \geq 2d+2$  as  $d \geq 3$ . For its right-hand side, we use (5.16)-(5.19). We obtain

$$(6.7) \quad \begin{aligned} M_j \xi_j + \tilde{f}_j(\xi, \eta) + U_j(M\xi, N\eta) &= f_j(\xi, \eta) + Df_j(\xi, \eta)(U, V) \\ &\quad + A_j(\xi, \eta) + (M_j^0 + DM_j^0(\eta U + \xi V + UV))(\xi_j + U_j) + O(2d+2), \end{aligned}$$

where  $M, N, M^0, N^0$  are evaluated at  $\xi\eta$  and  $U, V$  are evaluated at  $(\xi, \eta)$ .

Since  $\tilde{f}(M(\xi\eta)\xi, N(\xi\eta)\eta) = O(|\xi, \eta|^{2d})$ , then (6.1)-(6.2) follow from (5.26)-(5.27), where by Definition 5.1

$$\mathcal{U}_{j,PQ}^*(\cdot; 0) = \mathcal{V}_{j,PQ}^*(\cdot; 0) = 0.$$

Next, we refine (5.28) to verify the remaining assertions. We recall from (5.18) the remainders

$$A_j(\xi, \eta) = R_2 M_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) + R_2 f_j(\xi, \eta; U, V).$$

Here by (5.12), we have the Taylor remainder formula

$$R_2 f(x; y) = 2 \int_0^1 (1-t) \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha dt.$$

Since  $\text{ord}(U, V) \geq d$ ,  $\text{ord}(f, g) \geq d$ , and  $d \geq 4$ , then  $A_j$ , defined by (5.18), satisfies

$$A_j(\xi, \eta) = O(|(\xi, \eta)|^{2d+2}).$$

Recall that  $f_j(\xi, \eta)$  and  $U_j(\xi, \eta)$  do not contain terms of the form  $\xi_j \xi^P \eta^P$ , while  $g_j(\xi, \eta)$  and  $V_j(\xi, \eta)$  do not contain terms of the form  $\eta_j \xi^P \eta^P$ . Comparing both sides of (6.7) for coefficients of  $\xi_j \xi^P \eta^P$  with  $|P| = d-1$ , we get (6.4).

Assume now that (6.5) holds. Assume that  $i \neq j$ . Then  $D_i M_j^0(\xi\eta) = O(|\xi\eta|)$ . We see that  $D_i M_j^0(\xi\eta) \eta_i U_i(\xi, \eta)$  and  $D_i M_j^0(\xi\eta) \xi_i V_i(\xi, \eta)$  do not contain terms of  $\xi^P \eta^P$ , and

$$D_i M_j^0(\xi\eta) \xi_i U_i(\xi, \eta) V_i(\xi, \eta) = O(2d+3).$$

Since  $(\tilde{f}, \tilde{g}) \in \mathcal{C}_2^c(S)$  and  $(\tilde{f}, \tilde{g}) = O(2d)$ , then  $\tilde{f}_j(M(\xi\eta)\xi, N(\xi\eta)\eta)$  does not contain terms  $\xi^P \eta^P \xi_j$  for  $2|P|+1 = 2d+1$ . Now (6.6) follows from a direction computation.  $\square$

Set  $|\delta_N(\mu)| := \max \{|\nu| : \nu \in \delta_N(\mu)\}$  for

$$\delta_N(\mu) = \bigcup_{j=1}^p \left\{ \mu_j, \mu_j^{-1}, \frac{1}{\mu^P - \mu_j} : P \in \mathbf{Z}^p, P \neq e_j, |P| \leq N \right\}.$$

**Definition 6.3.** We say that  $\mu^{P^*-Q^*} - \mu_j$  and  $\mu^{Q^*-P^*} - \mu_j^{-1}$  are small divisors of *height*  $N$ , if there exists a partition

$$\bigcup_{i=1}^p \left\{ |\mu^{P-Q} - \mu_i| : P, Q \in \mathbf{N}^p, |P| + |Q| \leq N, \mu^{P-Q} \neq \mu_i \right\} = S_N^0 \cup S_N^1$$

with  $|\mu^{P^*-Q^*} - \mu_j| \in S_N^0$  and  $S_N^1 \neq \emptyset$  such that

$$\max S_N^0 < C \min S_N^0, \quad \max S_N^0 < (\min S_N^1)^{L_N} < 1.$$

Here  $C$  depends only on an upper bound of  $|\mu|$  and  $|\mu|^{-1}$  and

$$L_N \geq N.$$

If  $|\mu^{P_*-Q_*} - \mu_j|$  is in  $S_N^0$  and if  $P_*, Q_* \in \mathbf{N}^p$ , we call  $|P_* - Q_*|$  the *degree* of the small divisors  $\mu^{P_*-Q_*} - \mu_j$  and  $\mu^{Q_*-P_*} - \mu_j^{-1}$ .

To avoid confusion, let us call  $\mu^{P_*-Q_*} - \mu_j$  that appear in  $S_N^0$  the *exceptional* small divisors. These small divisors have been used by Cremer [Cr28] and Siegel [Si41]. The degree and height play different roles in computation. The height serves as the maximum degree of all small divisors that need to be considered in computation.

Roughly speaking, the quantities in  $S_N^0$  are comparable but they are much smaller than the ones in  $S_N^1$ . We will construct  $\mu$  for any prescribed sequence of positive integers  $L_N$  so that

$$\max S_N^0 < (\min S_N^1)^{L_N} < 1$$

for a subsequence  $N = N_k$  tending to  $\infty$ . Furthermore, to use the small divisors we will identify all exceptional small divisors of height  $2N_k + 1$  and all degrees of the exceptional small divisors with  $N_k$  being the smallest.

We start with the following lemma which gives us small divisors that decay as rapidly as we wish.

**Lemma 6.4.** *Let  $L_k$  be a strictly increasing sequence of positive integers. There exist a real number  $\nu \in (0, 1/2)$  and a sequence  $(p_k, q_k) \in \mathbf{N}^2$  such that  $e, 1, \nu$  are linearly independent over  $\mathbf{Q}$ , and*

$$(6.8) \quad |q_k \nu - p_k - e| \leq \Delta(p_k, q_k)^{L_{p_k+q_k}},$$

$$(6.9) \quad \Delta(p_k, q_k) = \min \left\{ \frac{1}{2}, |q\nu - p - re| : 0 < |r| + |q| < 3(q_k + 1), \right. \\ \left. (p, q, r) \neq 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1) \right\}.$$

*Proof.* We consider two increasing sequences  $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty$  of positive integers, which are to be chosen. For  $k = 1, 2, \dots$ , we set

$$\nu = \nu_k + \nu'_k, \quad \nu_k = \sum_{\ell=1}^k \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!}, \quad \nu'_k = \sum_{\ell>k} \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!}, \\ q_k = m_k!.$$

We choose  $m_k > (m_\ell)!(n_\ell!)$  for  $k > \ell$  and decompose

$$q_k \nu = p_k + e_k + e'_k, \\ p_k = m_k! \nu_{k-1} \in \mathbf{N}, \quad e_k = \sum_{\ell=0}^{n_k} \frac{1}{k!}, \quad e'_k = m_k! \nu'_k.$$

We have  $e'_k < m_k! \sum_{\ell > k} \frac{e}{m_\ell!}$  and

$$(6.10) \quad \begin{aligned} q_k \nu &= p_k + e + e'_k - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!}, \\ |q_k \nu - p_k - e| &\leq m_k! \nu'_k + \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!} < \{12(3(q_k+1)^3)!\}^{-L_{p_k+q_k}}. \end{aligned}$$

Here  $(6.10)_k$  is achieved by choosing  $(m_2, n_1), \dots, (m_{k+1}, n_k)$  successively. Clearly we can get  $0 < \nu < 1/2$  if  $m_1$  is sufficiently large.

Next, we want to show that  $re + p + q\nu \neq 0$  for all integers  $p, q, r$  with  $(p, q, r) \neq (0, 0, 0)$ . Otherwise, we rewrite  $-m_k!p = m_k!(q\nu + re)$  as

$$-m_k!p = qp_k + r \sum_{j=0}^{m_k} \frac{m_k!}{j!} + qe + q \left( e'_k - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!} \right) + r \sum_{j>m_k} \frac{m_k!}{j!}.$$

The left-hand side is an integer. On the right-hand side, the first two terms are integers,  $qe$  is a fixed irrational number, and the rest terms tend to 0 as  $k \rightarrow \infty$ . We get a contradiction.

To verify (6.8), we need to show that for each tuple  $(p, q, r)$  satisfying (6.9),

$$(6.11) \quad |q\nu - p - re| \geq |q_k \nu - p_k - e|^{\frac{1}{L_{p_k+q_k}}}.$$

We first note the following elementary inequality

$$(6.12) \quad |p + qe| \geq \frac{1}{(q-1)!} \min \left\{ 3 - e, \frac{1}{q+1} \right\}, \quad p, q \in \mathbf{Z}, \quad q \geq 1.$$

Indeed, the inequality holds for  $q = 1$ . For  $q \geq 2$  we have  $q!e = m + \epsilon$  with  $m \in \mathbf{N}$  and

$$\epsilon := \sum_{k=q+1}^{\infty} \frac{q!}{k!} > \frac{1}{q+1}.$$

Furthermore,  $1 - \epsilon > 1 - \frac{2}{q+1} = \frac{q-1}{q+1}$  as

$$\epsilon < \frac{1}{q+1} + \sum_{k \geq q+2} \frac{1}{k(k-1)} = \frac{2}{q+1}.$$

We may assume that  $q \geq 0$ . If  $q = 0$ , then  $|r| < 3q_k + 3$  by condition in (6.9) and hence  $|p + re| \geq \frac{1}{(3q_k+4)!}$  by (6.12). Now (6.11) follows from (6.10). Assume that  $q > 0$ . We have

$$(6.13) \quad \begin{aligned} |-q\nu + p + re| &\geq \left| -q \frac{p_k + e}{q_k} + p + re \right| - q \frac{|e + p_k - q_k \nu|}{q_k} \\ &= \left| \frac{q_k p - qp_k}{q_k} + \frac{rq_k - q}{q_k} e \right| - q \frac{|e + p_k - q_k \nu|}{q_k}. \end{aligned}$$

We first verify that  $q_k p - qp_k$  and  $q - rq_k$  do not vanish simultaneously. Assume that both are zero. Then  $(p, q, r) = r(p_k, q_k, 1)$ . Thus  $|r| \neq 1, 2$ , and  $|r| \geq 3$  by conditions in (6.9);

we obtain  $|r| + |q| \geq 3(|q_k| + 1)$ , a contradiction. Therefore, either  $q_k p - q p_k$  or  $r q_k - q$  is not zero. By (6.12) and (6.13),

$$\begin{aligned} |-q\nu + p + re| &\geq \frac{1}{q_k} \cdot \frac{1}{3} \cdot \frac{1}{(|rq_k - q| + 1)!} - q \frac{|e + p_k - q_k \nu|}{q_k} \\ &\geq \frac{1}{(3q_k + 4)^2!} - 4|e + p_k - q_k \nu|. \end{aligned}$$

Using (6.10) twice, we obtain the next two inequalities:

$$|-q\nu + p + re| \geq \frac{1}{2} \left\{ (3q_k + 4)^2! \right\}^{-1} \geq |p_k + e - q_k \nu|^{\frac{1}{L_{p_k + q_k}}}.$$

The two ends give us (6.11).  $\square$

We now reformulate the above lemma as follows.

**Lemma 6.5.** *Let  $L_k$  be a strictly increasing sequence of positive integers. Let  $\nu \in (0, 1/2)$ , and let  $p_k$  and  $q_k$  be positive integers as in Lemma 6.4. Set  $(\mu_1, \mu_2, \mu_3) := (e^{-1}, e^\nu, e^e)$ . Then*

$$(6.14) \quad |\mu^{P_k} - \mu_3| \leq (C\Delta^*(P_k))^{L_{|P_k|}}, \quad P_k = (p_k, q_k, 0),$$

$$(6.15) \quad \Delta^*(P_k) = \min_j \left\{ |\mu^R - \mu_j| : R \in \mathbf{Z}^3, |R| \leq 2(q_k + p_k) + 1, \right. \\ \left. R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \right\}.$$

Here  $C$  does not depend on  $k$ . Moreover, all exceptional small divisors of height  $2|P_k| + 1$  have degree at least  $|P_k|$ . Moreover,  $\mu^{P_k} - \mu_3$  is the only exceptional small divisor of degree  $|P_k|$  and height  $2|P_k| + 1$ .

In the definition of  $\Delta^*(P_k)$ , equivalently we require that

$$R \neq P_k, R_k^1, R_k^2, R_k^3$$

with  $R_k^1 := -P_k + 2e_3$ ,  $R_k^2 := 2P_k - e_3$ , and  $R_k^3 := -2P_k + 3e_3$ . Note that  $|R_k^1| = |P_k| + 2$ ,  $|R_k^2| = 2|P_k| + 1$ , and  $|R_k^3| = 2|P_k| + 3$  are bigger than  $|P_k|$ , i.e. the degree of the exceptional small divisor  $\mu^{P_k} - \mu_3$ . Each  $\mu^{R_k^i} - \mu_3$  is a small divisor comparable with  $\mu^{P_k} - \mu_3$ . Finally,  $\Delta^*(P_k)$  tends to zero as  $|P_k| \rightarrow \infty$ . Let us set  $N := 2|P_k| + 1$ , and

$$\begin{aligned} S_N^0 &:= \left\{ |\mu^{P_k} - \mu_3|, |\mu^{R_k^1} - \mu_3|, |\mu^{R_k^2} - \mu_3|, |\mu^{R_k^3} - \mu_3| \right\}, \\ S_N^1 &:= \bigcup_j \left\{ |\mu^R - \mu_j| : R \in \mathbf{Z}^3, |R| \leq 2(q_k + p_k) + 1, \right. \\ &\quad \left. R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \right\}. \end{aligned}$$

This implies that the last paragraph of Lemma 6.5 holds when the  $L_N$  in Definition 6.3, denoted it by  $L'_N$ , takes the value  $L'_N = L_{2N+1}$ , while  $L_N$  is prescribed in Lemma 6.5.

*Proof.* By Lemma 6.4, we find a real number  $\nu \in (0, 1/2)$  and positive integers  $p_k, q_k$  such that  $e, 1, \nu$  are linearly independent over  $\mathbf{Q}$  and

$$(6.16) \quad \begin{aligned} |q_k \nu - e - p_k| &\leq \Delta(p_k, q_k)^{L|P_k|}, \\ \Delta(p_k, q_k) &= \min \{|q\nu - re - p| : 0 < |r| + |q| < 3(q_k + 1), \\ &\quad (p, q, r) \neq 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1)\}. \end{aligned}$$

Note that  $\mu_1, \mu_2, \mu_3$  are positive and non-resonant. We have

$$|\mu^{P_k} - \mu_3| = |\mu_3| \cdot |e^{q_k \nu - p_k - e} - 1|.$$

Let  $\nu^* := (-1, \nu, e)$ . If  $|R \cdot \nu^* - \nu_j^*| < 2$ , then by the intermediate value theorem

$$e^{-2} |\mu_j| |R \cdot \nu^* - \nu_j^*| \leq |\mu^R - \mu_j| \leq e^2 |\mu_j| |R \cdot \nu^* - \nu_j^*|.$$

If  $R \cdot \nu^* - \nu_j^* > 2$  or  $R \cdot \nu^* - \nu_j^* < -2$ , we have

$$|\mu^R - \mu_j| \geq e^{-2} |\mu_j|.$$

Thus, we can restate the properties of  $\nu^*$  as follows:

$$\begin{aligned} |\mu^{-(p_k, q_k, 0)} - \mu_3| &\leq C'(C' \tilde{\Delta}(p_k, q_k))^{L|P_k|}, \\ \tilde{\Delta}(p_k, q_k) &= \min \{|\mu^{(p, q, r)} - 1| : 0 < |r| + |q| < 3(q_k + 1), \\ &\quad (p, q, r) \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1)\}. \end{aligned}$$

Recall that  $0 < \nu < 1/2$ . By (6.16), we have  $|q_k \nu - e - p_k| < 1$ . Since  $p_k, q_k$  are positive, then  $p_k < \nu q_k < q_k/2$ . Assume that  $|\mu^R - \mu_j| = \Delta^*(P_k)$ ,  $|R| \leq 2(p_k + q_k) + 1$ , and

$$R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1).$$

Set  $R' := R - e_j$  and  $(p, q, r) := R'$ . Then  $\Delta^*(P_k) = |\mu_j| |\mu^{R'} - 1|$ . Also,  $|r| + |q| \leq |R'| \leq |R| + 1 \leq 2(p_k + q_k) + 2 \leq q_k + 2q_k + 2 < 3(q_k + 1)$ . This shows that  $|\mu^{Q'} - 1| \geq \tilde{\Delta}(p_k, q_k)$ . We obtain  $\Delta^*(P_k) \geq \mu_j \tilde{\Delta}(p, q_k)$ . We have verified (6.14). For the remaining assertions, see the remark following the lemma.  $\square$

In the above we have retained  $\mu_j > 0$  which are sufficient to realize  $\mu_1, \mu_2, \mu_3, \mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}$  as eigenvalues of  $\sigma$  for an elliptic complex tangent. Indeed, with  $0 < \mu_1 < 1$ , interchanging  $\xi_1$  and  $\eta_1$  preserves  $\rho$  and changes the  $(\xi_1, \eta_1)$  components of  $\sigma$  into  $(\mu_1^{-1} \xi_1, \mu_1 \eta_1)$ .

We are ready to prove Theorem 1.4, which is restated here:

**Theorem 6.6.** *There exists a non-resonant elliptic real analytic 3-submanifold  $M$  in  $\mathbf{C}^6$  such that  $M$  admits the maximum number of deck transformations and all Poincaré-Dulac normal forms of the  $\sigma$  associated to  $M$  are divergent.*

*Proof.* We will not construct the real analytic submanifold  $M$  directly. Instead, we will construct a family of involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  so that all Poincaré-Dulac normal forms of  $\sigma$  are divergent. By the realization in Proposition 2.1, we get the desired submanifold.

We first give an outline of the proof. To prove the theorem, we first deal with the associated  $\sigma$  and its normal form  $\tilde{\sigma}$ , which belongs to the centralizer of  $S$ , the linear part of  $\sigma$  at the origin. Thus  $\sigma^*$  has the form

$$\sigma^*: \xi' = M(\xi\eta)\xi, \quad \eta' = N(\xi\eta)\eta.$$

We assume that  $\log M$  is tangent to identity at the origin. We then normalize  $\sigma^*$  into the normal form  $\hat{\sigma}$  stated in Theorem 5.6 (i). (In Lemma 6.1 we take  $F = \log M$  and  $\hat{F} = \log \hat{M}$ .) We will show that  $\hat{\sigma}$  is divergent if  $\sigma$  is well chosen. By Theorem 5.6 (iii), all normal forms of  $\sigma$  in the centralizer of  $S$  are divergent. To get  $\sigma^*$ , we use the normalization of Proposition 5.2 (i). To get  $\hat{\sigma}$ , we normalize further using Lemma 5.5. To find a divergent  $\hat{\sigma}$ , we need to tie the normalizations of two formal normal forms together, by keeping track of the small divisors in the two normalizations.

We will start with our initial pair of involutions  $\{\tau_1^0, \tau_2^0\}$  satisfying  $\tau_2^0 = \rho\tau_1^0\rho$  such that  $\sigma^0$  is a third order perturbation of  $S$ . We require that  $\tau_1^0$  be the composition of  $\tau_{11}^0, \dots, \tau_{1p}^0$ . The latter can be realized by a real analytic submanifold by using Proposition 2.1. We will then perform a sequence of holomorphic changes of coordinates  $\varphi_k$  such that  $\tau_1^k = \varphi_k\tau_1^{k-1}\varphi_k^{-1}$ ,  $\tau_2^k = \rho\tau_1^k\rho$ , and  $\sigma^k = \tau_1^k\tau_2^k$ . By abuse of notation,  $\tau_i^k, \sigma^k$ , etc. do not stand for iterating the maps  $k$  times. Each  $\varphi_k$  is tangent to the identity to order  $d_k$ . For a suitable choice of  $\varphi_k$ , we want to show that the coefficients of order  $d_k$  of the normal form of  $\sigma^k$  increase rapidly to the effect that the coefficients of the normal form of the limit mapping  $\sigma^\infty$  increase rapidly too. Here we will use the exceptional small divisors to achieve the rapid growth of the coefficients of the normal forms. Roughly speaking, the latter requires us to keep track the rapid growth for a sequence of coefficients in the normal form in a sequence of two-step normalizations. Recall from Lemma 6.1 that if we have

$$\sigma: \xi'_j = M_j(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = N_j(\xi\eta)\eta_j + g_j(\xi, \eta), \quad 1 \leq j \leq p.$$

with  $\text{ord}(f, g) \geq d \geq 4$  and  $(f, g) \in \mathcal{C}_2^c(S)$ , then there is a polynomial mapping  $\Psi: \xi' = \xi + \hat{U}(\xi, \eta), \eta' = \eta + V(\xi, \eta)$  in  $\mathcal{C}^c(S)$  that has order  $d$  and degree at most  $2d - 1$  such that for the new mapping

$$\hat{\sigma} := \Psi^{-1}\sigma\Psi: \xi'_j = \hat{M}_j(\xi\eta)\xi_j + \hat{f}_j(\xi, \eta), \quad \eta'_j = \hat{N}_j(\xi\eta)\eta_j + \hat{g}_j(\xi, \eta),$$

the coefficients of  $M_j(\xi\eta)\xi_j$  of degree  $2d + 1$  have the form

$$(6.17) \quad \hat{M}_{j,P} = M_{j,P} + \mu_j \left\{ 2(\hat{U}_j\hat{V}_j)_{PP} + (\hat{U}_j^2)_{(P+e_j)(P-e_j)} \right\} + \{Df_j(\hat{U}, \hat{V})\}_{(P+e_j)P}.$$

It is crucial that for suitable multi-indices, *both*  $\hat{U}_j, \hat{V}_j$  contain the exceptional small divisors of degree  $d$  as formulated in Lemma 6.5 (see also Definition 6.3). Although  $Df_j(\hat{U}, \hat{V})$  contains (exceptional) small divisors, they can only appear at most once in each term, provided  $f_j$  contains no small divisor of degree  $d$ . The formula (6.17) appears as simple as it is, it requires that  $\sigma$  has been normalized to degree  $d$ . To achieve such a  $\sigma$ , we need to use a preliminary change of coordinates  $\Phi: \xi' = \xi + U(\xi, \eta), \eta' = \eta + V(\xi, \eta)$  via polynomials of degree less than  $d$ . The  $\Phi$  depends only on small divisors of degree  $< d$ , but none of them are exceptionally small. Therefore, by composing  $\Psi\Phi$ , we obtain (6.17) where small divisors of degree  $< d$  are absorbed into terms  $M_{j,P}$  and the products of two exceptional small divisors in (6.17), if they exist, dominate the other terms in  $\hat{M}_{j,P}$ . Of course, we need to apply a sequence of transformations  $\Phi, \Psi$  and we should leave the coefficients of a certain degree unchanged in the process once they become large, which are possible by Corollary 5.4.

We now present the proof. Let  $\sigma^0 = \tau_1^0 \tau_2^0$ ,  $\tau_2^0 = \rho \tau_1^0 \rho$ , and

$$\begin{aligned} \tau_1^0: \xi_j' &= \Lambda_{1j}^0(\xi\eta)\eta_j, & \eta_j' &= (\Lambda_{1j}^0(\xi\eta))^{-1}\xi_j, \\ \sigma^0: \xi_j' &= (\Lambda_{1j}^0(\xi\eta))^2\xi_j, & \eta_j' &= (\Lambda_{1j}^0(\xi\eta))^{-2}(\xi\eta)\eta_j. \end{aligned}$$

Since we consider the elliptic case, we require that  $(\Lambda_{1j}^0(\xi\eta))^2 = \mu_j e^{\xi_j \eta_j}$ . So  $\zeta \rightarrow (\Lambda_1^0)^2(\zeta)$  is biholomorphic. Recall that  $\sigma^0$  can be realized by  $\{\tau_{11}^0, \dots, \tau_{1p}^0, \rho\}$ . We will take

$$(6.18) \quad \varphi_k: \xi_j' = (\xi - h^{(k)}(\xi), \eta), \quad \text{ord } h^{(k)} = d_k > 3,$$

$$(6.19) \quad d_k \geq 2d_{k-1}, \quad |h_p^{(k)}| \leq 1.$$

We will also choose each  $h_j^{(k)}(\xi)$  to have one monomial only. Let  $\Delta_r := \Delta_r^3$  denote the polydisc of radius  $r$ . Let  $\|\cdot\|$  be the sup norm on  $\mathbf{C}^3$ . Let  $H^{(k)}(\xi) = \xi - h^{(k)}(\xi)$  and we first verify that  $H_k = H^{(k)} \circ \dots \circ H^{(1)}$  converges to a holomorphic function on the polydisc  $\Delta_{r_1}$  for  $r_1 > 0$  sufficiently small; consequently,  $\varphi_k \circ \dots \circ \varphi_1$  converges to a germ of holomorphic map  $\varphi^\infty$  at the origin. Note that  $H^{(k)}$  sends  $\Delta_{r_k}$  into  $\Delta_{r_{k+1}}$  for  $r_{k+1} = r_k + r_k^{d_k}$ . We want to show that when  $r_1$  is sufficiently small,

$$(6.20) \quad r_k \leq s_k := (2 - \frac{1}{k})r_1.$$

It holds for  $k = 1$ . Let us show that  $r_{k+1}/r_k - 1 \leq \theta_k := s_{k+1}/s_k - 1$ , i.e.

$$r_k^{d_k-1} \leq \theta_k = \frac{1}{(k+1)(2k-1)}.$$

We have  $(2r_1)^{d_k-1} \leq (2r_1)^k$  when  $0 < r_1 < 1/2$ . Fix  $r_1$  sufficiently small such that  $(2r_1)^k < \frac{1}{(k+1)(2k-1)}$  for all  $k$ . By induction, we obtain (6.20) for all  $k$ . In particular, we have  $\|h^{(k)}(\xi)\| \leq \|\xi\| + \|H^{(k)}(\xi)\| \leq 2r_{k+1}$  for  $\|\xi\| < r_k$ . To show the convergence of  $H_k$ , we write  $H_k - H_{k-1}(\xi) = -h^{(k)} \circ H_{k-1}$ . By the Schwarz lemma, we obtain

$$\|h^{(k)} \circ H_{k-1}(\xi)\| \leq \frac{2r_{k+1}}{r_1^{d_k}} \|\xi\|^{d_k}, \quad \|\xi\| < r_1.$$

Note that the above estimate is uniform under conditions (6.18)-(6.19). Therefore,  $H_k$  converges to a holomorphic function on  $\|\xi\| < r_1$ .

Throughout the proof, we make initial assumptions that  $d_k$  and  $h^{(k)}$  satisfy (6.18)-(6.19),  $e^{-1} \leq \mu_j \leq e^e$ , and  $\mu^Q \neq 1$  for  $Q \in \mathbf{Z}^3$  with  $Q \neq 0$ . Set  $\sigma^k = \tau_1^k \tau_2^k$ ,  $\tau_2^k = \rho \tau_1^k \rho$ , and

$$\tau_1^k = \varphi_k \tau_1^{k-1} \varphi_k^{-1}.$$

We want  $\sigma^k$  not to be holomorphically equivalent to  $\sigma^{k-1}$ . Thus we have chosen a  $\varphi_k$  that does not commute with  $\rho$  in general. Note that  $\sigma^k$  is still generated by a real analytic submanifold; indeed, when  $\tau_i^{k-1} = \tau_{i1}^{k-1} \dots \tau_{ip}^{k-1}$  and  $\tau_{2j}^{k-1} = \rho \tau_{1j}^{k-1} \rho$ , we still have the same identities if the superscript  $k-1$  is replaced by  $k$  and  $\tau_{1j}^k$  equals  $\varphi_k \tau_{1j}^{k-1} \varphi_k^{-1}$ . It is clear that  $\sigma^k = \sigma^{k-1} + O(d_k)$ . As power series, we have

$$(6.21) \quad \sigma^\ell = \sigma^{k-1} + O(d_k), \quad k \leq \ell \leq \infty.$$



Note that as limits in convergence,  $\tau_{ij}^\infty = \lim_{k \rightarrow \infty} \tau_{ij}^k$ ,  $\tau_i^\infty = \lim_{k \rightarrow \infty} \tau_i^k$  and  $\sigma^\infty = \lim_{k \rightarrow \infty} \sigma^k$  satisfy

$$\tau_{2j}^\infty = \rho \tau_{1j}^\infty \rho, \quad \tau_i^\infty = \tau_{i1}^\infty \cdots \tau_{ip}^\infty, \quad \tau_2^\infty = \rho \tau_1^\infty \rho, \quad \sigma^\infty = \tau_1^\infty \tau_2^\infty.$$

Of course,  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  satisfies all the conditions that ensure it can be realized by a real analytic submanifold.

We know that  $\sigma^\infty$  does not have a unique normal form in the centralizer  $S$ . Therefore, we will choose a procedure that arrives at a unique formal normal form in  $S$ . We show that this unique normal form is divergent; and hence by Theorem 5.6 (iii) any normal form of  $\sigma$  that is in the centralizer of  $S$  must diverge.

We now describe the procedure. For a formal mapping  $F$ , we have a unique decomposition

$$F = NF + N^c F, \quad NF \in \mathcal{C}(S), \quad N^c F \in \mathcal{C}^c(S).$$

Set  $\hat{\sigma}_0^\infty = \sigma^\infty$ . For  $k = 0, 1, \dots$ , we take a normalized polynomial map  $\Phi_k \in \mathcal{C}_2^c(S)$  of degree less than  $d_k$  such that  $\sigma_k^\infty := \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$  is normalized up to degree  $d_k - 1$ . Specifically, we require that

$$\deg \Phi_k \leq d_k - 1, \quad \Phi_k \in \mathcal{C}^c(S); \quad N^c \sigma_k^\infty(\xi, \eta) = O(d_k).$$

Take a normalized polynomial map  $\Psi_{k+1}$  such that  $\Psi_{k+1}$  and  $\hat{\sigma}_{k+1}^\infty := \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$  satisfy

$$\deg \Psi_{k+1} \leq 2d_k - 1; \quad \Psi_{k+1} \in \mathcal{C}_2^c(S), \quad N^c \hat{\sigma}_{k+1}^\infty = O(2d_k).$$

We can repeat this for  $k = 0, 1, \dots$ . Thus we apply two sequences of normalization as follows

$$\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^\infty \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

We will show that  $\Psi_{k+1} = I + O(d_k)$  and  $\Phi_k = I + O(2d_{k-1})$ . This shows that the sequence  $\Phi_0 \Psi_1 \cdots \Phi_k \Psi_{k+1}$  defines a formal biholomorphic mapping  $\Phi$  so that

$$(6.22) \quad \hat{\sigma}^\infty := \Phi^{-1} \sigma^\infty \Phi$$

is in a normal form. Finally, we need to combine the above normalization with the normalization for the unique normal form in Lemma 5.5. We will show that the unique normal form diverges.

Let us recall previous results to show that  $\Phi_k, \Psi_{k+1}$  are uniquely determined. Set

$$(6.23) \quad \hat{\sigma}_k^\infty: \begin{cases} \xi' = \hat{M}^{(k)}(\xi\eta)\xi + \hat{f}^{(k)}(\xi, \eta), \\ \eta' = \hat{N}^{(k)}(\xi\eta)\eta + \hat{g}^{(k)}(\xi, \eta), \end{cases}$$

$$(6.24) \quad (\hat{f}^{(k)}, \hat{g}^{(k)}) \in \mathcal{C}_2^c(S).$$

Recall that  $\hat{\sigma}_0 = \sigma^\infty$ . Assume that we have achieved

$$(6.25) \quad (\hat{f}^{(k)}, \hat{g}^{(k)}) = O(2d_{k-1}).$$

Here we take  $d_{-1} = 2$  so that (6.23)-(6.25) hold for  $k = 0$ . By Proposition 5.2, there is a unique normalized mapping  $\tilde{\Phi}_k$  that transforms  $\hat{\sigma}_k^\infty$  into a normal form. We denote by  $\Phi_k$

the truncated polynomial mapping of  $\tilde{\Phi}_k$  of degree  $d_k - 1$ . We write

$$\begin{aligned}\Phi_k: \xi' &= \xi + U^{(k)}(\xi, \eta), \quad \eta' = \eta + V^{(k)}(\xi, \eta), \\ (U^{(k)}, V^{(k)}) &= O(2), \quad \deg(U^{(k)}, V^{(k)}) \leq d_k - 1.\end{aligned}$$

By Corollary 5.3,  $\Phi_k$  satisfies

$$\begin{aligned}(6.26) \quad \sigma_k^\infty &= \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k: \begin{cases} \xi' = M^{(k)}(\xi\eta)\xi + f^{(k)}(\xi, \eta), \\ \eta' = N^{(k)}(\xi\eta)\eta + g^{(k)}(\xi, \eta), \end{cases} \\ (f^{(k)}, g^{(k)}) &\in \mathcal{C}_2^c(S), \quad \text{ord}(f^{(k)}, g^{(k)}) \geq d_k.\end{aligned}$$

In fact, by (5.26)-(5.27) (or (5.24)-(5.25)), we have

$$(6.27) \quad U_{j,PQ}^{(k)} = (\mu^{P-Q} - \mu_j)^{-1} \left\{ \hat{f}_{j,PQ}^{(k)} + \mathcal{U}_{j,PQ}(\delta_{d-1}, \{\hat{M}^{(k)}, \hat{N}^{(k)}\}_{[\frac{d-1}{2}]}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{d-1}) \right\},$$

$$(6.28) \quad V_{j,QP}^{(k)} = (\mu^{Q-P} - \mu_j^{-1})^{-1} \left\{ \hat{g}_{j,QP}^{(k)} + \mathcal{V}_{j,QP}(\delta_{d-1}, \{\hat{M}^{(k)}, \hat{N}^{(k)}\}_{[\frac{d-1}{2}]}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{d-1}) \right\},$$

for  $|P| + |Q| = d < d_k$  and  $\mu^{P-Q} \neq \mu_j$ . By (5.28)-(5.29) (or (5.22)-(5.23)), we have

$$(6.29) \quad M_P^{(k)} = \hat{M}_P^{(k)} + \mathcal{M}_P(\delta_{2|P|-1}, \{\hat{M}^{(k)}, \hat{N}^{(k)}\}_{|P|-1}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{2|P|-1}),$$

$$(6.30) \quad N_P^{(k)} = \hat{N}_P^{(k)} + \mathcal{N}_P(\delta_{2|P|-1}, \{\hat{M}^{(k)}, \hat{N}^{(k)}\}_{|P|-1}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{2|P|-1})$$

for  $2|P| - 1 < d_k$ . Recall that  $\mathcal{U}_{j,PQ}$ ,  $\mathcal{V}_{j,QP}$ ,  $\mathcal{M}_{j,P}$ , and  $\mathcal{N}_{j,P}$  are universal polynomials in their variables. In notation defined by Definition 5.1,

$$\mathcal{U}_{j,PQ}(\bullet; 0) = \mathcal{V}_{j,QP}(\bullet; 0) = 0, \quad \mathcal{M}_P(\bullet; 0) = \mathcal{N}_P(\bullet; 0) = 0.$$

We apply (6.27)-(6.28) for  $d < 2d_{k-1} \leq d_k$  and (6.29)-(6.30) for  $2|P| - 1 < 2d_{k-1} \leq d_k$  to obtain

$$(6.31) \quad \Phi_k - I = (U^{(k)}, V^{(k)}) = O(2d_{k-1}),$$

$$(6.32) \quad M_P^{(k)} = \hat{M}_P^{(k)}, \quad N_P^{(k)} = \hat{N}_P^{(k)}, \quad |P| \leq d_{k-1}.$$

In fact, by Corollary 5.4, the above holds for  $|P| < 2d_{k-1} - 1$ .

By Lemma 6.1, there is a unique normalized polynomial mapping

$$\begin{aligned}\Psi_{k+1}(\xi, \eta) &= (\xi + \hat{U}^{(k+1)}(\xi, \eta), \eta + \hat{V}^{(k+1)}(\xi, \eta)), \\ (\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) &\in \mathcal{C}_2^c(S),\end{aligned}$$

$$(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) = O(2), \quad \deg(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \leq 2d_k - 1$$

such that  $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \Phi_k^{-1} \sigma_k^\infty \Phi_k \Psi_{k+1}$  satisfies the following:

$$\begin{aligned}\hat{\sigma}_{k+1}^\infty: \xi' &= \hat{M}^{(k+1)}(\xi\eta)\xi + \hat{f}^{(k+1)}, \quad \eta' = \hat{N}^{(k+1)}(\xi\eta)\eta + \hat{g}^{(k+1)}, \\ (\hat{f}^{(k+1)}, \hat{g}^{(k+1)}) &\in \mathcal{C}_2^c(S), \quad \text{ord}(\hat{f}^{(k+1)}, \hat{g}^{(k+1)}) \geq 2d_k.\end{aligned}$$

By (6.1)-(6.2), we know that

$$(6.33) \quad \hat{U}_{j,PQ}^{(k+1)} = (\mu^{P-Q} - \mu_j)^{-1} \left\{ \hat{f}_{j,PQ}^{(k)} + \mathcal{U}_{j,PQ}^*(\delta_{\ell-1}, \{M^{(k)}, N^{(k)}\}_{[\frac{\ell-1}{2}]}; \{f^{(k)}, g^{(k)}\}_{\ell-1}) \right\},$$

$$(6.34) \quad \hat{V}_{j,QP}^{(k+1)} = (\mu^{Q-P} - \mu_j^{-1})^{-1} \left\{ \hat{g}_{j,QP}^{(k)} + \mathcal{V}_{j,QP}^*(\delta_{\ell-1}, \{M^{(k)}, N^{(k)}\}_{[\frac{\ell-1}{2}]}; \{f^{(k)}, g^{(k)}\}_{\ell-1}) \right\},$$

for  $d_k \leq |P| + |Q| = \ell \leq 2d_k - 1$  and  $\mu^{P-Q} \neq \mu_j$ . Recall that  $\mathcal{U}_{j,PQ}^*$  and  $\mathcal{V}_{j,QP}^*$  are universal polynomials in their variables. In notation defined by Definition 5.1,  $\mathcal{U}_{j,PQ}^*(\cdot; 0) = \mathcal{V}_{j,QP}^*(\cdot; 0) = 0$ . Thus

$$(6.35) \quad \Psi_{k+1} - I = (\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) = O(d_k),$$

$$(6.36) \quad \hat{U}_{j,PQ}^{(k+1)} = \frac{f_{j,PQ}^{(k)}}{\mu^{P-Q} - \mu_j}, \quad \hat{V}_{j,QP}^{(k+1)} = \frac{g_{j,QP}^{(k)}}{\mu^{Q-P} - \mu_j^{-1}}, \quad |P| + |Q| = d_k.$$

Here  $\mu^{P-Q} \neq \mu_j$ . By (6.3)-(6.6), we have

$$(6.37) \quad \hat{M}_{j,P'}^{(k+1)} = M_{j,P'}^{(k)}, \quad |P'| < d_k - 1;$$

$$(6.38) \quad \hat{M}_{j,P'}^{(k+1)} = M_{j,P}^{(k)} + \left\{ Df_j^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \right\}_{(P+e_j)P}, \quad |P_k| = d_k - 1;$$

$$(6.39) \quad \begin{aligned} \hat{M}_{j,P}^{(k+1)} &= M_{j,P}^{(k)} + \mu_j \left\{ 2(\hat{U}_j^{(k+1)} \hat{V}_j^{(k+1)})_{PP} + ((\hat{U}_j^{(k+1)})^2)_{(P+e_j)(P-e_j)} \right\} \\ &\quad + \left\{ Df_j^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \right\}_{(P+e_j)P}, \quad |P| = d_k. \end{aligned}$$

As stated in Corollary 5.4, the coefficients of  $\hat{M}_j^{(k+1)}(\xi\eta)\xi_j$  of degree  $2d_k + 1$  do not depend on the coefficients of  $f^{(k)}, g^{(k)}$  of degree  $\geq 2d_k$ , provided  $(f^{(k)}, g^{(k)}) = O(d_k)$  is in  $\mathcal{C}_2^c(S)$  as it is assumed.

Next, we need to estimate the size of coefficients of  $M^{(k)}$  that appear in (6.37)-(6.39). Recall that we apply two sequences of normalization. We have

$$\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^\infty \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

Thus,  $M^{(k)}, N^{(k)}$  depend only on  $\sigma^\infty, \Phi_0, \Psi_1, \Phi_1, \dots, \Psi_{k-1}, \Phi_k$ .

Recall that if  $u_1, \dots, u_m$  are power series, then  $\{u_1, \dots, u_m\}_d$  denotes the set of their coefficients of degree at most  $d$ , and  $|\{u_1, \dots, u_m\}_d|$  denotes the sup norm. We need some crude estimates on the growth of Taylor coefficients. If  $F = I + f$  and  $f = O(2)$  is a map in formal power series, then (5.1)-(5.3) imply

$$(6.40) \quad |\{F^{-1}\}_m| \leq (2 + |\{f\}_m|)^{\ell_m},$$

$$|\{G \circ F\}_m| \leq (2 + |\{f, G\}_m|)^{\ell_m},$$

$$(6.41) \quad |\{F^{-1} \circ G \circ F\}_m| \leq (2 + |\{f, G\}_m|)^{\ell_m},$$

In general, if  $F_j$  are formal mappings of  $\mathbf{C}^n$  that are tangent to the identity, then

$$|\{F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k\}_m| \leq (2 + |\{F_1, \dots, F_k, G\}_m|)^{\ell_{m,k}}, \quad 1 \leq k < \infty$$

In particular, if  $F_j = I + O(j)$  for  $j = 1, \dots, m$ , then for any  $k \geq |P| := m$  we have

$$(F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k)_P = (F_m^{-1} \cdots F_1^{-1} G F_1 \cdots F_m)_P,$$

$$|\{F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k\}_m| \leq (2 + |\{F_1, \dots, F_m, G\}_m|)^{\ell_m}, \quad 1 \leq k \leq \infty.$$

We may take  $\ell'_m$  by  $\ell_m$ , while  $\ell_m$  depends only on  $m$ . We have similar estimates for  $F_k \cdots F_1 G F_1^{-1} \cdots F_k^{-1}$ . Recall that  $1/\sqrt{2} < \lambda_j < e^{e/2} < 4$ . Using  $\tau_{1j}^k = \varphi_k \tau_{1j}^{k-1} \varphi_k^{-1} =$

$\varphi_k \dots \varphi_1 \tau_{1j}^0 \varphi_1^{-1} \dots \varphi_k^{-1}$  and hence  $\tau_1^k = \varphi_k \dots \varphi_1 \tau_1^0 \varphi_1^{-1} \dots \varphi_k^{-1}$  and  $\sigma_k = \tau_1^k(\rho \tau_2^k \rho)$ , we obtain  $|\{\tau_1^k\}_m| \leq (2 + |\{\tau_1^0, \varphi_1, \dots, \varphi_m\}_m|)^{\ell'_m} \leq 6^{\ell'_m}$  and  $|\{\sigma^k\}_m| \leq (8^{\ell'_m})^{\ell_m}$ . Thus we obtain

$$(6.42) \quad |\{\sigma^k\}_m| \leq 8^{\ell_m \ell'_m}, \quad |\{\sigma^\infty\}_m| \leq 8^{\ell_m \ell'_m}.$$

Here we have used (6.18)-(6.19).

For simplicity, let  $\delta_i$  denote  $\delta_i(\mu)$ . Inductively, let us show that for  $k = 0, 1, \dots$ ,

$$(6.43) \quad |\{\hat{M}^{(k)}, \hat{N}^{(k)}\}_P| \leq |\delta_{d_{k-1}-1}|^{L_m}, \quad m = 2|P| + 1 < 2d_{k-1} - 1,$$

$$(6.44) \quad |\{\hat{\sigma}^\infty\}_{PQ}| \leq |\delta_{2d_{k-1}-1}|^{L_m}, \quad m = |P| + |Q| \geq 2d_{k-1} - 1.$$

$$(6.45) \quad |M_{j,P}^{(k)}| + |N_{j,P}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = 2|P| + 1,$$

$$(6.46) \quad |f_{j,PQ}^{(k)}| + |g_{j,PQ}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = |P| + |Q| \geq d_k.$$

Note that the last inequalities are equivalent to  $|\{\sigma_k^\infty\}_m| \leq |\delta_{d_k-1}|^{L_m}$ . Here and in what follows  $L_m$  does not depend on the choices of  $\mu_j, d_k, h^{(k)}$  which satisfy the initial conditions, i.e.  $1/e \leq \mu_j \leq e^e$  and (6.18)-(6.19) but are arbitrary otherwise. However, it suffices to find constants  $L_{m,k}$  replacing  $L_m$  and depending on  $k$  such that (6.43)-(6.46) hold. Indeed, by (6.31) and (6.35) we have  $\Psi_{k+1} = I + O(d_k)$  and  $\Phi_k = I + O(2d_{k-1})$ . Since  $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$  and  $\sigma_k^\infty = \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$ , then

$$\hat{\sigma}_{k+1}^\infty = \hat{\sigma}_k^\infty + O(2d_{k-1}), \quad \sigma_{k+1}^\infty = \sigma_k^\infty + O(d_k)$$

as  $d_k \geq 2d_{k-1}$ . Since  $d_k$  increases to  $\infty$  with  $k$ , then (6.43)-(6.46) with  $L_{m,k}$  in place of  $L_m$  imply that they also hold for

$$L_m = \min_k \max\{L_{m,1}, \dots, L_{m,k} : (\hat{\sigma}_\ell^\infty - \hat{\sigma}_k^\infty, \sigma_\ell^\infty - \sigma_k^\infty) = O(m), \ell > k\}.$$

Therefore, in the following the dependence of  $L_m$  on  $k$  will not be indicated. The estimates (6.43)-(6.44) hold trivially for  $\hat{\sigma}_0^\infty = \sigma^\infty$ ,  $k = 0$  and  $d_{-1} = 2$  by (6.21) and (6.42). Assuming (6.43)-(6.44), we want to verify (6.45)-(6.46). We also want to verify (6.43)-(6.44) when  $k$  is replaced by  $k+1$ .

The  $\Phi_k = I + (U^{(k)}, V^{(k)})$  is a polynomial mapping. Its degree is at most  $d_k - 1$  and its coefficients are polynomials in  $\{\hat{\sigma}_k\}_{d_k-1}$  and  $\delta_{d_k-1}$ ; see (6.27)-(6.28). Hence

$$(6.47) \quad |U_{j,PQ}^{(k)}| + |V_{j,PQ}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = |P| + |Q|.$$

Applying (6.41) to  $\sigma_k^\infty = \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$ , we obtain (6.45)-(6.46) from (6.43)-(6.44). Here we use that fact that since  $d_k \geq 2d_{k-1}$ , the small divisors in  $\delta_{2d_{k-1}-1}$  appear in  $\delta_{d_k-1}$  too. To obtain (6.43)-(6.44) when  $k$  is replaced by  $k+1$ , we note that  $\Psi_{k+1}$  is a polynomial map that has degree at most  $2d_k - 1$  and the coefficients of degree  $m$  bounded by  $\delta_{2d_k-1}^{L_m}$ ; see (6.33)-(6.34). This shows that

$$(6.48) \quad |\hat{U}_{j,PQ}^{(k+1)}| + |\hat{V}_{j,PQ}^{(k+1)}| \leq |\delta_{2d_k-1}|^{L_m}, \quad |P| + |Q| = m.$$

We then obtain (6.43)-(6.44) when  $k$  is replaced by  $k+1$  for  $\hat{\sigma}_{k+1}^\infty$  by applying (6.41) to  $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$  and by using (6.45)-(6.46) for  $\sigma_k^\infty$  and (6.48) for  $\Psi_{k+1}$ .

Let us summarize the above computation for  $\hat{\sigma}^\infty$  defined by (6.22). We know that  $\hat{\sigma}^\infty$  is the unique power series such that  $\hat{\sigma}^\infty - \hat{\sigma}_k^\infty = O(d_k)$  for all  $k$ , and  $\hat{\sigma}^\infty$  is a formal normal

form of  $\sigma^\infty$ . Let us write

$$\hat{\sigma}^\infty: \begin{cases} \xi' = \hat{M}^\infty(\xi\eta)\xi, \\ \eta' = \hat{N}^\infty(\xi\eta)\eta. \end{cases}$$

Let  $|P| \leq d_k$ . By (6.32), we get  $\hat{M}_P^{(k+1)} = M_P^{(k+1)}$ ; by (6.37) in which  $k$  is replaced by  $k+1$ , we get  $\hat{M}_P^{(k+2)} = M_P^{(k+1)}$  as  $|P| \leq d_k < d_{k+1} - 1$ . Therefore,

$$(6.49) \quad \hat{M}_P^\infty = \hat{M}_P^{(k+1)}, \quad |P| \leq d_k.$$

For  $|P| < d_k - 1$ , (6.37) says that  $\hat{M}_{j,P}^{(k+1)} = M_{j,P}^{(k)}$ ; by (6.45) that holds for any  $P$ , we obtain

$$(6.50) \quad |\hat{M}_P^\infty| = |\hat{M}_{j,P}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = 2|P| + 1, \quad |P| < d_k - 1,$$

$$(6.51) \quad |\hat{M}_{j,P}^\infty| \leq |\delta_{d_k-1}|^{L_m}(1 + |\delta_{d_k}|), \quad m = 2|P| + 1 = 2d_k - 1.$$

We have verified (6.43)-(6.46). The sequence  $L_m$  depend only on

$$(6.52) \quad m = d_k + 1, \quad d_k, d_{k-1}, \dots, d_0, \quad d_j \geq 2d_{j-1}, \quad d_j > 3.$$

To obtain rapid increase of coefficients of  $\hat{M}_{j,P}^{(k+1)}$ , we want to use both small divisors hidden in  $\hat{U}_{j,PQ}^{(k)}$  and  $\hat{V}_{j,QP}^{(k)}$  in (6.39). Therefore, if  $M_{j,P}^{(k)}$  is already sufficiently large for  $|P| = d_k$  that will be specified later, we take  $\varphi_k$  to be the identity, i.e.  $\tau_1^k = \tau_1^{k-1}$ . Otherwise, we need to achieve it by choosing

$$\tau_1^k = \varphi_k \tau_1^{k-1} \varphi_k^{-1}.$$

Therefore, we examine the effect of a coordinate change by  $\varphi_k$  on these coefficients.

Recall that we are in the elliptic case. We have  $\rho(\xi, \eta) = (\bar{\eta}, \bar{\xi})$  and  $\tau_2^k = \rho \tau_1^k \rho$ . Recall that

$$\varphi_k: \xi'_j = (\xi - h^{(k)}(\xi), \eta), \quad \text{ord } h^{(k)} = d_k > 3.$$

By a simple computation, we obtain

$$\begin{aligned} \tau_1^k(\xi, \eta) &= \tau_1^{k-1}(\xi, \eta) + (-h^{(k)}(\lambda\eta), \lambda^{-1}h^{(k)}(\xi)) + O(|(\xi, \eta)|^{d_k+1}), \\ \tau_2^k(\xi, \eta) &= \tau_2^{k-1}(\xi, \eta) + (\lambda^{-1}\overline{h^{(k)}}(\eta), -\overline{h^{(k)}}(\lambda\xi)) + O(|(\xi, \eta)|^{d_k+1}). \end{aligned}$$

Then we have

$$(6.53) \quad \begin{aligned} \sigma^k &= \sigma^{k-1} + (r^{(k)}, s^{(k)}) + O(d_k + 1); \\ r^{(k)}(\xi, \eta) &= -\lambda \overline{h^{(k)}}(\lambda\xi) - h^{(k)}(\lambda^2\xi), \\ s^{(k)}(\xi, \eta) &= \lambda^{-2}\overline{h^{(k)}}(\eta) + \lambda^{-1}h^{(k)}(\lambda^{-1}\eta). \end{aligned}$$

Since  $\sigma^k$  converges to  $\sigma^\infty$ , from (6.53) it follows that

$$(6.54) \quad \sigma^\infty = \sigma^{k-1} + (r^{(k)}, s^{(k)}) + O(d_k + 1).$$

For  $|P| + |Q| = d_k$ , we have

$$\begin{aligned} r_{j,PQ}^{(k)} &= \left\{ -\lambda_j \overline{h_j^{(k)}}(\lambda\xi) - h_j^{(k)}(\lambda^2\xi) \right\}_{PQ}, \\ s_{j,QP}^{(k)} &= \left\{ \lambda_j^{-2} \overline{h_j^{(k)}}(\eta) + \lambda_j^{-1} h_j^{(k)}(\lambda^{-1}\eta) \right\}_{QP}. \end{aligned}$$

We obtain

$$(6.55) \quad r_{j,P0}^{(k)} = -\lambda^{P+e_j} \overline{h_{j,P}^{(k)}} - \lambda^{2P} h_{j,P}^{(k)},$$

$$(6.56) \quad s_{j,0P}^{(k)} = \lambda_j^{-2} \overline{h_{j,P}^{(k)}} + \lambda^{-P-e_j} h_{j,P}^{(k)}, \quad |P| = d_k,$$

$$(6.57) \quad r_{j,PQ}^{(k)} = s_{j,QP}^{(k)} = 0, \quad |P| + |Q| = d_k, \quad Q \neq 0.$$

The above computation is actually sufficient to construct a divergent normal form  $\tilde{\sigma} \in \mathcal{C}(S)$ . To show that all normal forms of  $\sigma$  in  $\mathcal{C}(S)$  are divergent, We need to related it to the normal form  $\hat{\sigma}$  in Theorem 5.6, which is unique. This requires us to keep track of the small divisors in the normalization procedure in the proof of Lemma 5.5.

Recall that  $F^{(k+1)} = \log \hat{M}^{(k+1)}$  is defined by

$$(6.58) \quad F_j^{(k+1)}(\zeta) = \log(\mu_j^{-1} \hat{M}_j^{(k+1)}(\zeta)) = \zeta_j + a_j^{(k+1)}(\zeta), \quad 1 \leq j \leq 3.$$

We also have  $F^\infty = \log \hat{M}^\infty$  with  $F_j^\infty(\zeta) = \zeta_j + a_j^\infty(\zeta)$ . Then by (6.49),

$$(6.59) \quad a_{j,P}^\infty = a_{j,P}^{(k+1)}, \quad |P| \leq d_k.$$

By (6.58) and  $\log(1+x) = x + \frac{x^2}{2} + O(3)$ , we have

$$(6.60) \quad a_{j,P}^{(k+1)}(\zeta) = \mu_j^{-1} \hat{M}_{j,P}^{(k+1)} + \mu_j^{-1} \hat{M}_{j,P-e_j}^{(k+1)} + \mathcal{A}_{j,P}(\{\hat{M}_j^{(k+1)}\}_{|P|-2}), \quad |P| > 1.$$

By (6.50)-(6.51), we estimate the last two terms as follows

$$(6.61) \quad |\mathcal{A}_{j,P}(\{\hat{M}_j^{(k+1)}\}_{|P|-2})| \leq |\delta_{d_k-1}|^{L_m^* L_m}, \quad |P| = d_k, \quad m = 2|P| + 1,$$

$$(6.62) \quad |\hat{M}_{j,Q}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m} (1 + |\delta_{d_k}|), \quad m = 2|Q| + 1 = 2d_k - 1.$$

Here  $L_m^* \geq 1$  is independent of  $k$  and depends only on the degrees of the polynomials  $\mathcal{A}_{j,P}$ . Recall from the formula (5.36) that  $F^{(k+1)}, F^\infty$  have the normal forms  $\hat{F}^{(k+1)} = I + \hat{a}^{(k+1)}$  and  $\hat{F}^\infty = I + \hat{a}^\infty$ , respectively. The coefficients of  $\hat{a}_{j,Q}^{(k+1)}$  and  $\hat{a}_{j,Q}^\infty$  are zero, except the ones given by

$$\begin{aligned} \hat{a}_{j,Q}^{(k+1)} &= a_{j,Q}^{(k+1)} - \{Da_j^{(k+1)} \cdot a^{(k+1)}\}_Q + \mathcal{B}_{j,Q}(\{a^{(k+1)}\}_{|Q|-2}), \\ \hat{a}_{j,Q}^\infty &= a_{j,Q}^\infty - \{Da_j^\infty \cdot a^{(\infty)}\}_Q + \mathcal{B}_{j,Q}(\{a^{(\infty)}\}_{|Q|-2}), \end{aligned}$$

for  $Q = (q_1, \dots, q_p)$ ,  $q_j = 0$ , and  $|Q| > 1$ . Derived from the same normalization, the  $\mathcal{B}_{j,Q}$  in both formulae stands for the same polynomial and independent of  $k$ . Hence  $\hat{a}_{j,P}^{(\infty)} = \hat{a}_P^{(k+1)}$  for  $|P| \leq d_k$ , by (6.59). Combining (6.39) and (6.49) yields

$$(6.63) \quad \begin{aligned} \hat{a}_{3,Q}^\infty &= \hat{a}_{3,Q}^{(k+1)} = 2(\hat{U}_3^{(k+1)} \hat{V}_3^{(k+1)})_{QQ} + ((\hat{U}_3^{(k+1)})^2)_{(Q+e_3)(Q-e_3)} + \mu_3^{-1} M_{3,Q}^{(k)} \\ &\quad + \mu_3^{-1} \{Df_3^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)})\}_{(Q+e_3)P_k} + \mathcal{A}_Q(\{\hat{M}^{(k+1)}\}_{|Q|-2}) \\ &\quad + \mu_3^{-1} \hat{M}_{Q-e_3}^{(k+1)} - \{Da_j^{(k+1)} \cdot a^{(k+1)}\}_Q. \end{aligned}$$

The above formula holds for any  $Q$  with  $|Q| = d_k$ . To examine the effect of small divisors, we assume that

$$P_k = (p_k, q_k, 0), \quad |P_k| = d_k$$

are given by Lemma 6.5, so are  $\mu_1, \mu_2$ , and  $\mu_3$ . **However,  $P_k$  and  $\mu$  depend on a sequence  $L_m$  (to be renamed as  $L'_m$ ) in Lemma 6.5. We will determine the sequence  $L'_m$  and hence  $P_k$  and  $\mu_j$  later.**

Note that the second term in (6.63) is 0 as the third component of  $P_k - e_3$  is negative. We apply the above computation to the  $P_k$ . Taking a subsequence of  $P_k$  if necessary, we may assume that  $d_k \geq 2d_{k-1}$  and  $d_{k-1} > 3$  for all  $k \geq 1$ . The 4 exceptional small divisors of height  $2|P_k| + 1$  in (6.15) are

$$\mu^{P_k} - \mu_3, \quad \mu^{-P_k} - \mu_3^{-1}, \quad \mu^{2P_k - e_3} - \mu_3, \quad \mu^{-2P_k + e_3} - \mu_3^{-1}.$$

The last two cannot show up in  $\hat{a}_{3,P_k}^\infty$ , since their degree,  $2d_k + 1$ , is larger than the degrees of Taylor coefficients in  $\hat{a}_{3,P_k}$ . We have 3 products of the two exceptional small divisors of height  $2|P_k| + 1$  and degree  $|P_k|$ , which are

$$(\mu^{P_k} - \mu_3)(\mu^{-P_k} - \mu_3^{-1}), \quad (\mu^{P_k} - \mu_3)(\mu^{P_k} - \mu_3), \quad (\mu^{-P_k} - \mu_3^{-1})(\mu^{-P_k} - \mu_3^{-1}).$$

The first product, but none of the other two, appears in  $(\hat{U}_3^{(k+1)} \hat{V}_3^{(k+1)})_{P_k P_k}$ . The third term and  $f_3^{(k)}$  in  $\hat{a}_{3,P_k}^\infty$  do not contain exceptional small divisors of degree  $|P_k| = d_k > 2d_{k-1} - 1$ . Since  $f_3^{(k)} = O(d_k)$  by (6.26), the exceptional small divisors of height  $2|P_k| + 1$  can show up at most once in the fourth term of  $\hat{a}_{3,P_k}^\infty$ . Therefore, we arrive at

$$\begin{aligned} \hat{a}_{3,P_k}^\infty &= 2\hat{U}_{3,P_k 0}^{(k+1)} \hat{V}_{3,0 P_k}^{(k+1)} + \hat{\mathcal{A}}_{P_k}^1(\delta_{d_k-1}, \frac{1}{\mu^{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k}) \\ &\quad + \hat{\mathcal{A}}_{P_k}^2(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}) + \mu_3^{-1} M_{3,P_k}^{(k)} + \mathcal{A}_{P_k}(\{\hat{M}^{(k+1)}\}_{|P_k|-2}) \\ &\quad + \mu_3^{-1} M_{3,P_k-1}^{(k+1)} - \{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}, \\ \hat{\mathcal{A}}_{P_k}^1(\delta_{d_k-1}, \frac{1}{\mu^{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k}) &= (\hat{U}_{3,P_k 0}^{(k+1)}, \hat{V}_{3,0 P_k}^{(k+1)}) \cdot \hat{\mathcal{A}}_{P_k}^3(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}). \end{aligned}$$

Note that  $\hat{\mathcal{A}}_{P_k}^i$  and  $\mathcal{A}_{P_k}$  are polynomials independent of  $k$ . Set

$$m = 2d_k + 1.$$

In the following we can increase the value of  $L_m^*$  in (6.61) or when it reappears for a finite number of times such that the estimates involving  $L_m^*$  are valid for all  $k$ . By (6.45) and (6.61), we obtain  $|M_{3,P_k}^{(k)}| + |\mathcal{A}_{P_k}^3(\hat{M}^{(k+1)})_{|P_k|-2}| \leq \delta_{d_k-1}^{L_m^* L_m}$ . By (6.60), the smallest  $|Q|$  for which  $a_{i,Q}$  contains an exceptional small divisor in  $\delta_{d_k}$  is  $2|Q| + 1 = 2d_k - 1$ . Now,  $\{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}$  is a linear combination of products of two terms and at most one of the two terms contains an exceptional small divisor; if the both terms contain an exceptional small divisor, one term is  $a_{3,Q'}^{(k+1)}$  with  $2|Q'| + 1 \geq 2d_k - 1$ , while another is  $a_{i,Q''}^{(k+1)}$  with  $2|Q''| + 1 \geq 2d_k - 1$ . (Here  $Q' - e_i + Q'' = P_k$  and the  $i$ th component of  $Q'$  is positive.) Then  $d_k = |P_k| = |Q'| + |Q''| - 1 \geq 2d_k - 2$ , a contradiction. Therefore, by (6.60)-(6.62), we have

$$|\{a_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}| \leq |\delta_{d_k-1}|^{L_m^* L_m} (1 + |\delta_{d_k}|).$$

By (6.62), we also have  $|M_{j,P_k-1}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m}(1 + |\delta_{d_k}|)$ . Omitting the arguments in the polynomial functions, we obtain from (6.47)-(6.48), and (6.49) that

$$\begin{aligned} & |\hat{\mathcal{A}}_{P_k}^1| + |\hat{\mathcal{A}}_{P_k}^2| + |M_{3,P_k-1}^{(k+1)}| + |M_{3,P_k}^{(k)}| + |\mathcal{A}_{P_k}| + |\{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}| \\ & \leq \frac{|\delta_{d_k-1}(\mu)|^{L_m^* L_m}}{|\mu^{P_k} - \mu_3|}, \end{aligned}$$

for  $m = 2|P_k| + 1$  and a possibly larger  $L_m$ . We remark that although each term in the inequality depends on the choices of the sequences  $\mu_i, d_j, h^{(\ell)}$ , the  $L_m$  does not depend on the choices, provided that  $\mu_j, d_k, h^{(i)}$  satisfy our initial conditions. Therefore, we have

$$|\hat{a}_{3,P_k}^\infty| \geq 2|\hat{U}_{3,P_k,0}^{(k+1)} \hat{V}_{3,0P_k}^{(k+1)}| - |\delta_{d_k-1}(\mu)|^{L_{2|P_k|+1}^* L_{2|P_k|+1}} |\mu^{P_k} - \mu_3|^{-1}.$$

Recall that  $\sigma_k^\infty = \Phi_k^{-1} \Psi_{k-1}^{-1} \cdots \Phi_0^{-1} \sigma^\infty \Phi_0 \Psi_1 \cdots \Phi_k$ . Set

$$\tilde{\sigma}_k^\infty := \Phi_k^{-1} \Psi_{k-1}^{-1} \cdots \Phi_0^{-1} \sigma^{k-1} \Phi_0 \Psi_1 \cdots \Phi_k.$$

By (6.54), we get

$$(6.64) \quad \sigma_k^\infty = \tilde{\sigma}_k^\infty + (r^{(k)}, s^{(k)}) + O(d_k + 1).$$

Recall that  $\Phi_k$  depends only on coefficients of  $\hat{\sigma}_{k-1}^\infty = \Psi_{k-1}^{-1} \sigma_{k-2}^\infty \Psi_{k-1}$  of degree less than  $d_k$ , while  $\Psi_{k-1}$  depends only on coefficients of  $\sigma_{k-1}^\infty = \Phi_{k-1}^{-1} \hat{\sigma}_{k-1} \Phi_{k-1}$  of degree at most  $2d_{k-1} - 1$  which is less than  $d_k$  too. Therefore,  $\Phi_k, \Psi_{k-1}, \dots, \Phi_0$  depend only on coefficients of  $\sigma^\infty$  of degree less than  $d_k$ . On the other hand,  $\sigma^\infty = \sigma^{k-1} + O(d_k)$ . Therefore,  $\tilde{\sigma}_k^\infty$  depends only on  $\sigma^{k-1}$ , and hence it depends only on  $h^{(\ell)}$  for  $\ell < k$ . By (6.64), we can express

$$(6.65) \quad f_{j,PQ}^{(k)} = \tilde{f}_{j,PQ}^{(k)} + r_{j,PQ}^{(k)}, \quad g_{j,QP}^{(k)} = \tilde{g}_{j,QP}^{(k)} + s_{j,QP}^{(k)},$$

where  $|P| + |Q| = d_k$  and  $\tilde{f}_{j,PQ}^{(k)}, \tilde{g}_{j,QP}^{(k)}$  depend only on  $h^{(\ell)}$  for  $\ell < k$ . Collecting (6.36), (6.65), and (6.55)-(6.57), we obtain

$$|\hat{a}_{3,P_k}^\infty| \geq 2 \frac{|T_k|}{|\mu^{P_k} - \mu_3| |\mu^{-P_k} - \mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}^* L_{2d_k+1}}}{|\mu^{P_k} - \mu_3|}$$

with

$$\begin{aligned} T_k &= (-\lambda^{P_k+e_3} \overline{h_{3,P_k}^{(k)}} - \lambda^{2P_k} h_{3,P_k}^{(k)} + \tilde{f}_{3,P_k,0}^{(k-1)}) (\lambda_3^{-2} \overline{h_{3,P_k}^{(k)}} + \lambda^{-P_k-e_3} h_{3,P_k}^{(k)} + \tilde{g}_{3,0P_k}^{(k-1)}) \\ &= -\lambda^{2P_k-2e_3} (\lambda^{e_3-P_k} \overline{h_{3,P_k}^{(k)}} + h_{3,P_k}^{(k)} - \lambda^{-2P_k} \tilde{f}_{3,P_k,0}^{(k-1)}) (\lambda^{e_3-P_k} h_{3,P_k}^{(k)} + \overline{h_{3,P_k}^{(k)}} + \lambda_3^2 \tilde{g}_{3,0P_k}^{(k-1)}). \end{aligned}$$

Set  $\tilde{T}_k(h_{3,P_k}^{(k)}) := -\lambda^{2e_3-2P_k} T_k$ . We are ready to choose  $h_{3,P_k}^{(k)}$  to get a divergent normal form. We have  $|\lambda^{P_k-e_3} + 1| \geq 1$ . Then one of  $|\tilde{T}_k(0)|, |\tilde{T}_k(1)|, |\tilde{T}_k(-1)|$  is at least  $1/4$ ; otherwise, we would have

$$2|\lambda^{P_k-e_3} + 1|^2 = |\tilde{T}_k(1) + \tilde{T}_k(-1) - 2\tilde{T}_k(0)| < 1,$$

which is a contradiction. This shows that by taking  $h_{3,P_k}^{(k)}$  to be one of  $0, 1, -1$ , we have achieved

$$|T_k| \geq \frac{1}{4} \mu^{P_k-e_3}.$$



Therefore,

$$(6.66) \quad |\hat{a}_{3,P_k}^\infty| \geq \frac{\mu^{P_k - e_3}}{2|\mu^{P_k} - \mu_3||\mu^{-P_k} - \mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}^* L_{2d_k+1}}}{|\mu^{P_k} - \mu_3|}.$$

Recall that  $\mu_3 = e^e$ . If  $|\mu^{P_k} - \mu_3| < 1$  then  $1/2 < \mu^{P_k - e_3} < 2$ . The above inequality implies

$$(6.67) \quad |\hat{a}_{3,P_k}^\infty| \geq \frac{\mu^{2P_k}}{4|\mu^{P_k} - \mu_3|^2},$$

provided

$$|\mu^{P_k} - \mu_3| \leq \frac{1}{4} |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}^* L_{2d_k+1}}, \quad |P_k| = d_k.$$

For the last inequality to hold, it suffices have

$$(6.68) \quad |\mu^{P_k} - \mu_3| \leq |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}^* L_{2d_k+1} - 1}, \quad |\delta_{d_k-1}(\mu)|^{-1} < 1/4.$$

When (6.67), we still have (6.66). Thus we have derived universal constants  $L_{2d_k+1}, L_{2d_k+1}^*$  for any  $P_k = (p_k, q_k, 0)$  as long as  $|P_k| = d_k > 3$ . The sequence  $L_m^*, L_m$  do not depend on the choice of  $\lambda$  and they are independent of  $k$ ; however it depends on  $d_0, d_1, \dots, d_k$  as described in (6.52). Let us denote the constants  $L_{2d_k+1}, L_{2d_k+1}^*$  in (6.68) respectively by  $(L_{2d_k+1}(d_0, \dots, d_k), L_{2d_k+1}^*(d_0, \dots, d_k))$ . We now remove the dependence of  $L_m$  on the partition  $d_0, \dots, d_k$  and define  $L_m$  for  $m > 7$  as follows. For each  $m > 7$ , define

$$\begin{aligned} \mathcal{D}_m &= \{(d_0, \dots, d_k) : 3 < d_0 \leq d_1/2 \leq \dots \leq d_k/2^k, 2d_k + 1 \leq m, k = 0, 1, \dots\}, \\ L'_N &= N + 2 \max\{(L_{2d_k+1} L_{2d_k+1}^*)(d_0, \dots, d_k) : (d_0, \dots, d_k) \in \mathcal{D}_{2N+1}\}. \end{aligned}$$

Let us apply Lemma 6.5 to the sequence  $L'_N$ . Therefore, there exist  $\mu$  and a sequence of  $P_k = (p_k, q_k, 0)$  satisfying  $|\mu^{P_k} - \mu_3| \leq (C\Delta^*(P_k))^{L'_{|P_k|}}$ . Taking a subsequence if necessary, we may assume that  $d_k = |P_k| \geq 2d_{k-1}$  and  $d_k > 3$ . Thus

$$\begin{aligned} |\mu^{P_k} - \mu_3| &\leq (C\Delta^*(P_k))^{L'_{|P_k|}} \leq (\Delta^*(P_k)^{1/2})^{L'_{|P_k|}} \\ &\leq (\delta_{d_k-1}(\mu))^{-L'_{|P_k|}/2} \leq |\delta_{d_k-1}(\mu)|^{-L'_{2d_k+1}} \\ &\leq |\delta_{d_k-1}(\mu)|^{-L_{2D_K+1}^*(d_0, \dots, d_k) L_{2d_k+1}(d_0, \dots, d_k) - 1}, \end{aligned}$$

which gives us (6.68). Here the second inequality follows from  $C(\Delta^*(P_k))^{1/2} < 1$  when  $k$  is sufficiently large. The third inequality is obtained as follows. The definition of  $\Delta^*(P_k)$  and  $|P_k| = d_k$  imply that any small divisor in  $\delta_{d_k-1}(\mu)$  is contained in  $\Delta^*(P_k)$ . Also,  $\Delta^*(P_k) < \mu_i^{-1}$  for  $i = 1, 2, 3$  and  $k$  sufficiently large. Hence,  $\Delta^*(P_k) \leq \delta_{d_k-1}^{-1}(\mu)$ , which gives us the third inequality. We have that  $L_k \geq k$ . From (6.67) and (6.68) it follows that

$$|\hat{a}_{3,P_k}^\infty| > \delta_{d_k-1}^{d_k+1}(\mu) = \delta_{d_k-1}^{|P_k|+1}(\mu),$$

for  $k$  sufficiently large. As  $\delta_{d_k}(\mu) \rightarrow +\infty$ , this shows that the divergence of  $\hat{F}_3$  and the divergence of the normal form  $\hat{\sigma}$ .

As mentioned earlier, Theorem 5.6 (iii) implies that any normal form of  $\sigma$  that is in the centralizer of  $\hat{S}$  must diverge.  $\square$

## 7. A UNIQUE FORMAL NORMAL FORM OF A REAL SUBMANIFOLD

Recall that we consider submanifolds of which the complexifications admit the maximum number of deck transformations. The deck transformations of  $\pi_1$  are generated by  $\{\tau_{i1}, \dots, \tau_{ip}\}$ . We also set  $\tau_{2j} = \rho\tau_{1j}\rho$ . Each of  $\tau_{i1}, \dots, \tau_{ip}$  fixes a hypersurface and  $\tau_i = \tau_{i1} \cdots \tau_{ip}$  is the unique deck transformation of  $\pi_i$  whose set of fixed points has the smallest dimension. We first normalize the composition  $\sigma = \tau_1\tau_2$ . This normalization is reduced to two normal form problems. In Proposition 5.2 we obtain a transformation  $\Psi$  to transform  $\tau_1, \tau_2$ , and  $\sigma$  into

$$\begin{aligned}\tau_i^*: \xi'_j &= \Lambda_{ij}(\xi\eta)\eta_j, & \eta'_j &= \Lambda_{ij}^{-1}(\xi\eta)\xi_j, \\ \sigma^*: \xi'_j &= M_j(\xi\eta)\xi_j, & \eta'_j &= M_j^{-1}(\xi\eta)\eta_j, \quad 1 \leq j \leq p.\end{aligned}$$

Here  $\Lambda_{2j} = \Lambda_{1j}^{-1}$  and  $M_j = \Lambda_{1j}^2$  are power series in the product  $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$ . We also normalize the map  $M: \zeta \rightarrow M(\zeta)$  by a transformation  $\varphi$  which preserves all coordinate hyperplanes. This is the second normal form problem, which is solved formally in Theorem 5.6 under the condition on the normal form of  $\sigma$ , namely, that  $\log \hat{M}$  is tangent to the identity. This gives us a map  $\Psi_1$  which transforms  $\tau_1, \tau_2$ , and  $\sigma$  into  $\hat{\tau}_1, \hat{\tau}_2, \hat{\sigma}$  of the above form where  $\Lambda_{ij}$  and  $M_j$  become  $\hat{\Lambda}_{ij}, \hat{M}_j$ .

In this section, we derive a **unique formal normal form for  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  under the above condition on  $\log \hat{M}$** . In this case, we know from Theorem 5.6 that  $\mathcal{C}(\hat{\sigma})$  consists of only  $2^p$  dilatations

$$(7.1) \quad R_\epsilon: (\xi_j, \eta_j) \rightarrow (\epsilon_j \xi_j, \epsilon_j \eta_j), \quad \epsilon_j = \pm 1, \quad 1 \leq j \leq p.$$

We will consider two cases. In the first case, we impose no restriction on the linear parts of  $\{\tau_{ij}\}$  but the coordinate changes are restricted to mappings that are tangent to the identity. The second is for the family  $\{\tau_{ij}\}$  that arises from a higher order perturbation of a product quadric, while no restriction is imposed on the changes of coordinates. We will show that in both cases, if the normal form of  $\sigma$  can be achieved by a convergent transformation, the normal form of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  can be achieved by a convergent transformation too.

We now restrict our real submanifolds to some classes. First, we assume that  $\sigma$  and  $\tau_1, \tau_2$  are already in the normal form  $\hat{\sigma}$  and  $\hat{\tau}_1, \hat{\tau}_2$  such that

$$(7.2) \quad \hat{\tau}_i: \xi' = \hat{\Lambda}_i(\xi\eta)\eta, \quad \eta' = \hat{\Lambda}_i(\xi\eta)^{-1}\xi, \quad \hat{\Lambda}_2 = \hat{\Lambda}_1^{-1},$$

$$(7.3) \quad \hat{\sigma}: \xi' = \hat{M}(\xi\eta)\xi, \quad \eta' = \hat{M}(\xi\eta)^{-1}\eta, \quad \hat{M} = \hat{\Lambda}_1^2.$$

Let us start with the general situation without imposing the restriction on the linear part of  $\log M$ . Assume that  $\hat{\sigma}$  and  $\hat{\tau}_i$  are in the above forms. We want to describe  $\{\tau_{1j}, \rho\}$ . Let us start with the linear normal forms described in Lemma 3.5 or in Proposition 3.10. Recall that  $\mathbf{Z}_j = \text{diag}(1, \dots, -1, \dots, 1)$  with  $-1$  at the  $(p+j)$ -th place, and  $\mathbf{Z} := \mathbf{Z}_1 \cdots \mathbf{Z}_p$ . Let  $Z_j$  (resp.  $Z$ ) be the linear transformation with the matrix  $\mathbf{Z}_j$  (resp.  $\mathbf{Z}$ ). We also use notation

$$(7.4) \quad \mathbf{B}_* = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \quad \mathbf{E}_{\hat{\Lambda}_i} = \begin{pmatrix} \mathbf{I} & \hat{\Lambda}_i \\ -\hat{\Lambda}_i^{-1} & \mathbf{I} \end{pmatrix}.$$

Here  $\mathbf{B}$ , as well as  $\hat{\Lambda}_i$  given by (7.2), is a non-singular complex  $(p \times p)$  matrix. Assume that  $\mathbf{B}_1$  and  $\mathbf{B}_2$  are invertible  $p \times p$  matrices. Define

$$(7.5) \quad (B_i)_* : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow (\mathbf{B}_i)_* \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad E_{\hat{\Lambda}_i} : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{I} & \hat{\Lambda}_i(\xi\eta) \\ -\hat{\Lambda}_i^{-1}(\xi\eta) & \mathbf{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Let us assume that in suitable linear coordinates, the linear parts  $L\tau_{ij} = T_{ij}$  of two families of involutions  $\{\tau_{i1}, \dots, \tau_{ip}\}$  for  $i = 1, 2$  are given by

$$(7.6) \quad T_{ij} := E_{\Lambda_i, \mathbf{B}_i} \circ Z_j \circ E_{\Lambda_i, \mathbf{B}_i}^{-1},$$

$$(7.7) \quad E_{\Lambda_i, \mathbf{B}_i} := E_{\Lambda_i} \circ (B_i)_*, \quad \Lambda_i := \hat{\Lambda}_i(0).$$

Note that  $(B_i)_*$  commutes with  $Z$ . Also,  $E_{\hat{\Lambda}_i} \circ \hat{\tau}_i = Z \circ E_{\hat{\Lambda}_i}$ . We have the decomposition

$$(7.8) \quad \hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{ip},$$

$$(7.9) \quad E_{\hat{\Lambda}_i, \mathbf{B}_i} := E_{\hat{\Lambda}_i} \circ (B_i)_*, \quad \hat{\tau}_{ij} := E_{\hat{\Lambda}_i, \mathbf{B}_i} \circ Z_j \circ E_{\hat{\Lambda}_i, \mathbf{B}_i}^{-1}.$$

As before, we assume that  $S$  is non resonant. For real submanifolds, we still impose the reality condition  $\tau_{2j} = \rho\tau_{1j}\rho$  where  $\rho$  is given by (1.3). The following lemma describes a way to classify all involutions  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  provided that  $\sigma$  is in a normal form.

**Lemma 7.1.** *Let  $\{\tau_{1j}\}$  and  $\{\tau_{2j}\}$  be two families of formal holomorphic commuting involutions. Let  $\tau_i = \tau_{i1} \cdots \tau_{ip}$  and  $\sigma = \tau_1\tau_2$ . Suppose that*

$$\begin{aligned} \tau_i = \hat{\tau}_i : \xi'_j &= \hat{\Lambda}_{ij}(\xi\eta)\eta_j, & \eta'_j &= \hat{\Lambda}_{ij}(\xi\eta)^{-1}\xi_j; \\ \sigma = \hat{\sigma} : \xi'_j &= \hat{M}_j(\xi\eta)\xi_j, & \eta'_j &= \hat{M}_j(\xi\eta)^{-1}\eta_j \end{aligned}$$

with  $\hat{M}_j = \hat{\Lambda}_{1j}^2$  and  $\hat{M}_j(0) = \mu_j$ . Suppose that  $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$  satisfy the non-resonant condition (1.4). Assume further that the linear parts  $T_{ij}$  of  $\tau_{ij}$  are given by (7.6). Then we have the following :

- (i) For  $i = 1, 2$  there exists  $\Phi_i \in \mathcal{C}(\hat{\tau}_i)$ , tangent to the identity, such that  $\Phi_i^{-1}\tau_{ij}\Phi_i = \hat{\tau}_{ij}$  for  $1 \leq j \leq p$ .
- (ii) Let  $\{\tilde{\tau}_{1j}\}$  and  $\{\tilde{\tau}_{2j}\}$  be two families of formal holomorphic commuting involutions. Suppose that  $\tilde{\tau}_i = \hat{\tau}_i$  and  $\tilde{\sigma} = \hat{\sigma}$  and  $\tilde{\Phi}_i^{-1}\tilde{\tau}_{ij}\tilde{\Phi}_i = \hat{\tau}_{ij}$  with  $\tilde{\Phi}_i \in \mathcal{C}(\hat{\tau}_i)$  being tangent to the identity and

$$\hat{\tau}_{ij} = E_{\hat{\Lambda}_i, \tilde{\mathbf{B}}_i} \circ Z_j \circ E_{\hat{\Lambda}_i, \tilde{\mathbf{B}}_i}^{-1}.$$

Here for  $i = 1, 2$ , the matrix  $\tilde{\mathbf{B}}_i$  is non-singular. Then

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad i = 1, 2, \quad j = 1, \dots, p$$

if and only if there exist  $\Upsilon \in \mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$  and  $\Upsilon_i \in \mathcal{C}(\hat{\tau}_i)$  such that

$$(7.10) \quad \begin{aligned} \tilde{\Phi}_i &= \Upsilon^{-1} \circ \Phi_i \circ \Upsilon_i, \quad i = 1, 2, \\ \Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i &= \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p. \end{aligned}$$

Here each  $\nu_i$  is a permutation of  $\{1, \dots, p\}$ .

- (iii) Assume further that  $\tau_{2j} = \rho\tau_{1j}\rho$  with  $\rho$  being defined by (1.3). Define  $\hat{\tau}_{1j}$  by (7.8) and let  $\hat{\tau}_{2j} := \rho\hat{\tau}_{1j}\rho$ . Then we can choose  $\Phi_2 = \rho\Phi_1\rho$  for (i). Suppose that  $\tilde{\Phi}_2 = \rho\tilde{\Phi}_1\rho$  where  $\tilde{\Phi}_1$  is as in (ii). Then  $\{\tilde{\tau}_{1j}, \rho\}$  is equivalent to  $\{\tau_{1j}, \rho\}$  if and only if there exist  $\Upsilon_i, \nu_i$  with  $\nu_2 = \nu_1$ , and  $\Upsilon$  satisfying the conditions in (ii) and  $\Upsilon_2 = \rho\Upsilon_1\rho$ . The latter implies that  $\Upsilon\rho = \rho\Upsilon$ .

*Proof.* (i) Note that  $\hat{\tau}_{ij}$  is conjugate to  $Z_j$  via the map  $E_{\hat{\mathbf{A}}_i, \mathbf{B}_i}$ . Fix  $i$ . Each  $\hat{\tau}_{ij}$  is an involution and its set of fixed-point is a hypersurface. Furthermore,  $\text{Fix}(\tau_{11}), \dots, \text{Fix}(\tau_{1p})$  intersect transversally at the origin. By [GS15, Lemma 2.4], there exists a formal mapping  $\psi_i$  such that  $\psi_i^{-1}\tau_{ij}\psi_i = L\tau_{ij}$ . Now  $L\psi_i$  commutes with  $L\tau_{ij}$ . Replacing  $\psi_i$  by  $\psi_i(L\psi_i)^{-1}$ , we may assume that  $\psi_i$  is tangent to the identity. We also find a formal mapping  $\hat{\psi}_i$ , which is tangent to the identity, such that  $\hat{\psi}_i^{-1}\hat{\tau}_{ij}\hat{\psi}_i = L\hat{\tau}_{ij} = L\tau_{ij}$ . Then  $\Phi_1 = \psi_i\hat{\psi}_i^{-1}$  fulfills the requirements.

(ii) Suppose that

$$\tau_{ij} = \Phi_i\hat{\tau}_{ij}\Phi_i^{-1}, \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i\hat{\tau}_{ij}\tilde{\Phi}_i^{-1}.$$

Assume that there is a formal biholomorphic mapping  $\Upsilon$  that transforms  $\{\tau_{ij}\}$  into  $\{\tilde{\tau}_{ij}\}$  for  $i = 1, 2$ . Then

$$(7.11) \quad \Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

Here  $\nu_i$  is a permutation of  $\{1, \dots, p\}$ . Then

$$(7.12) \quad \hat{\tau}_i\Upsilon = \Upsilon\hat{\tau}_i, \quad \hat{\sigma}\Upsilon = \Upsilon\hat{\sigma}.$$

Set  $\Upsilon_i := \Phi_i^{-1}\Upsilon\tilde{\Phi}_i$ . We obtain

$$(7.13) \quad \Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p,$$

$$(7.14) \quad \tilde{\Phi}_i = \Upsilon^{-1}\Phi_i\Upsilon_i, \quad i = 1, 2.$$

Conversely, assume that (7.12)-(7.14) are valid. Then (7.11) holds as

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \Upsilon^{-1}\Phi_i\hat{\tau}_{ij}\Phi_i^{-1}\Upsilon = \tilde{\Phi}_i\Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i\tilde{\Phi}_i^{-1} = \tilde{\tau}_{i\nu_i(j)}.$$

(iii) Assume that we have the reality assumption  $\tau_{2j} = \rho\tau_{1j}\rho$  and  $\tilde{\tau}_{2j} = \rho\tilde{\tau}_{1j}\rho$ . As before, we take  $\Phi_1$ , tangent to the identity, such that  $\tau_{1j} = \Phi_1\hat{\tau}_{1j}\Phi_1^{-1}$ . Let  $\Phi_2 = \rho\Phi_1\rho$ . By  $\hat{\tau}_{2j} = \rho\hat{\tau}_{1j}\rho$ , we get  $\tau_{2j} = \rho\tau_{1j}\rho = \Phi_2\hat{\tau}_{2j}\Phi_2^{-1}$  for  $\nu_2 = \nu_1$ . Suppose that  $\tilde{\Phi}_i$  associated with  $\tilde{\tau}_{1j}$  and  $\rho$  satisfy the analogous properties. Suppose that  $\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}$  with  $\nu_2 = \nu_1$ , and  $\Upsilon\rho = \rho\Upsilon$ . Letting  $\Upsilon_1 = \Phi_1^{-1}\Upsilon\tilde{\Phi}_1$  we get  $\Upsilon_2 = \rho\Upsilon_1\rho$ . Conversely, if  $\Upsilon_1$  and  $\Upsilon_2$  satisfy  $\Upsilon_2 = \rho\Upsilon_1\rho$ , then

$$\rho\Upsilon\rho = \rho\Phi_1\Upsilon_1\tilde{\Phi}_1^{-1}\rho = \Phi_2\Upsilon_2\tilde{\Phi}_2^{-1} = \Upsilon.$$

This shows that  $\Upsilon$  satisfies the reality condition.  $\square$

Now we assume that  $\hat{F} = \log \hat{M}$  is tangent to the identity and is in the normal form (5.35). Recall the latter means that the  $j$ th component of  $\hat{F} - I$  is independent of the  $j$ th variable. We assume that the linear part  $T_{ij}$  of  $\tau_{ij}$  are given by (7.6), where the non-singular matrix  $\mathbf{B}$  is arbitrary. As mentioned earlier in this section, the group of formal biholomorphisms that preserve the form of  $\hat{\sigma}$  consists of only linear involutions  $R_\epsilon$  defined by (7.1). This restricts the holomorphic equivalence classes of the quadratic parts of  $M$ .

By Proposition 3.10, such quadrics are classified by a more restricted equivalence relation, namely,  $(\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2) \sim (\mathbf{B}_1, \mathbf{B}_2)$ , if and only if

$$\tilde{\mathbf{B}}_i = (\text{diag } \mathbf{a})^{-1} \mathbf{B}_i \text{diag}_{\nu_i} \mathbf{d}, \quad i = 1, 2.$$

To deal with a general situation, let us assume for the moment that  $\mathbf{B}_1, \mathbf{B}_2$  are arbitrary invertible matrices.

Using the normal form  $\{\hat{\tau}_1, \hat{\tau}_2\}$  and the matrices  $\mathbf{B}_1, \mathbf{B}_2$ , we first decompose  $\hat{\tau}_i = \hat{\tau}_{11} \cdots \hat{\tau}_{1p}$ . By Lemma 7.1 (i), we then find  $\Phi_i$  such that

$$\tau_{ij} = \Phi_i \hat{\tau}_{ij} \Phi_i^{-1}, \quad 1 \leq j \leq p.$$

For each  $i$ ,  $\Phi_i$  commutes with  $\hat{\tau}_i$ . It is within this family of  $\{\mathbf{B}_i, \Phi_i; i = 1, 2\}$  with  $\Phi_i \in \mathcal{C}(\hat{\tau}_i)$  for  $i = 1, 2$  that we will find a normal form for  $\{\tau_{ij}\}$ . When restricted to  $\tau_{2j} = \rho \tau_{1j} \rho$ , the classification of the real submanifolds is within the family of  $\{\tau_{1j}, \rho\}$  as described in Lemma 7.1 (iii).

From Lemma 7.1 (ii), the equivalence relation on  $\mathcal{C}(\hat{\tau}_i)$  is given by

$$\tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i, \quad i = 1, 2.$$

Here  $\Upsilon_i$  and  $\Upsilon$  satisfy

$$\Upsilon_i^{-1} \hat{\tau}_{ij} \Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p; \quad \Upsilon^{-1} \hat{\tau}_i \Upsilon = \hat{\tau}_i, \quad i = 1, 2.$$

We now construct a normal form for  $\{\tau_{ij}\}$  within the above family. Let us first use the centralizer of  $\mathcal{C}^c(Z_1, \dots, Z_p)$ , described in Lemma 4.4, to define the complement of the centralizer of the family of non-linear commuting involutions  $\{\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}\}$ . Recall that the mappings  $E_{\hat{\mathbf{A}}_i}$  and  $(B_i)_*$  are defined by (7.5). According to Lemma 4.4, we have the following.

**Lemma 7.2.** *Let  $i = 1, 2$ . Let  $\{\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}\}$  be given by (7.9). Then*

$$\begin{aligned} \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}) &= \left\{ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i} \circ \phi_0 \circ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i}^{-1} : \phi_0 \in \mathcal{C}(Z_1, \dots, Z_p) \right\}, \\ \mathcal{C}(\hat{\tau}_i) &= \left\{ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i} \circ \phi_0 \circ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i}^{-1} : \phi_0 \in \mathcal{C}(Z) \right\}. \end{aligned}$$

Set

$$\mathcal{C}^c(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}) := \left\{ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i} \circ \phi_1 \circ E_{\hat{\mathbf{A}}_i, \mathbf{B}_i}^{-1} : \phi_1 \in \mathcal{C}^c(Z_1, \dots, Z_p) \right\}.$$

Each formal biholomorphic mapping  $\psi$  admits a unique decomposition  $\psi_1 \psi_0^{-1}$  with

$$\psi_1 \in \mathcal{C}^c(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad \psi_0 \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}).$$

If  $\hat{\tau}_{ij}$  and  $\psi$  are convergent, then  $\psi_0, \psi_1$  are convergent. Assume further that  $\hat{\tau}_{2j} = \rho \hat{\tau}_{1j} \rho$  with  $\rho$  being given by (3.8). Then define  $\mathcal{C}^c(\hat{\tau}_{21}, \dots, \hat{\tau}_{2p}) = \{\rho \phi_1 \rho : \phi_1 \in \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})\}$ .

**Proposition 7.3.** *Let  $\hat{\tau}_i, \hat{\sigma}$  be given by (7.2)-(7.3) in which  $\log \hat{M}$  is in the formal normal form (5.35). Let  $\{\hat{\tau}_{ij}\}$  be given by (7.9). Suppose that*

$$(7.15) \quad \tau_{ij} = \Phi_i \hat{\tau}_{ij} \Phi_i^{-1}, \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i \hat{\tau}_{ij} \tilde{\Phi}_i^{-1} \quad 1 \leq j \leq p,$$

$$(7.16) \quad \Phi_i \in \mathcal{C}(\hat{\tau}_i), \quad \tilde{\Phi} \in \mathcal{C}(\hat{\tau}_i), \quad \tilde{\Phi}'_i(0) = \Phi'_i(0) = \mathbf{I}, \quad i = 1, 2.$$

Then  $\{\Upsilon^{-1}\tau_{ij}\Upsilon\} = \{\tilde{\tau}_{ij}\}$  for  $i = 1, 2$  and for some invertible  $\Upsilon \in \mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$ , if and only if there exist formal biholomorphisms  $\Upsilon, \Upsilon_1^*, \Upsilon_2^*$  such that

$$(7.17) \quad \Upsilon^{-1} \circ (B_i)_* \circ Z_j \circ (B_i)_*^{-1} \circ \Upsilon = (B_i)_* \circ Z_{\nu_i(j)} \circ (B_i)_*^{-1},$$

$$(7.18) \quad \tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i^* \Upsilon, \quad \Upsilon_i^* \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad i = 1, 2,$$

$$(7.19) \quad \Upsilon \hat{\sigma} \Upsilon^{-1} = \hat{\sigma},$$

where each  $\nu_i$  is a permutation of  $\{1, \dots, p\}$ . Assume further that  $\hat{\tau}_{2j} = \rho \hat{\tau}_{1j} \rho$  and  $\Phi_2 = \rho \Phi_1 \rho$  and  $\tilde{\Phi}_2 = \rho \tilde{\Phi}_1 \rho$ . We can take  $\Upsilon_2^* = \rho \Upsilon_1^* \rho$  and  $\nu_2 = \nu_1$ , if additionally

$$\Upsilon \rho = \rho \Upsilon.$$

*Proof.* Recall that

$$\tau_{ij} = \Phi_i \hat{\tau}_{ij} \Phi_i^{-1}, \quad \Phi_i \in \mathcal{C}(\hat{\tau}_i); \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i \hat{\tau}_{ij} \tilde{\Phi}_i^{-1}, \quad \tilde{\Phi}_i \in \mathcal{C}(\hat{\tau}_i).$$

Suppose that

$$(7.20) \quad \Upsilon^{-1} \tau_{ij} \Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

By Lemma 7.1, there are invertible  $\Upsilon_i$  such that

$$(7.21) \quad \Upsilon_i^{-1} \hat{\tau}_{ij} \Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p,$$

$$(7.22) \quad \tilde{\Phi}_i = \Upsilon^{-1} \circ \Phi_i \circ \Upsilon_i, \quad i = 1, 2.$$

Let us simplify the equivalence relation. By Theorem 5.6,  $\mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$  consists of  $2^p$  dilations  $\Upsilon$  of the form  $(\xi, \eta) \rightarrow (a\xi, a\eta)$  with  $a_j = \pm 1$ . Since  $\Phi_i, \tilde{\Phi}_i$  are tangent to the identity, then  $D\Upsilon_i(0)$  is diagonal because

$$L\Upsilon_i = \Upsilon.$$

Clearly,  $\Upsilon$  commutes with each non-linear transformation  $E_{\hat{\Lambda}_i}$ . Simplifying the linear parts of both sides of (7.21), we get

$$(7.23) \quad \Upsilon^{-1} \circ ((B_i)_* \circ Z_j \circ (B_i)_*^{-1}) \circ \Upsilon = (B_i)_* \circ Z_{\nu_i(j)} \circ (B_i)_*^{-1}.$$

From the commutativity of  $\Upsilon$  and  $E_{\hat{\Lambda}_i}$  again and the above identity, it follows that

$$(7.24) \quad \Upsilon^{-1} \circ \hat{\tau}_{ij} \circ \Upsilon = \hat{\tau}_{\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

Using (7.15) and (7.24), we can rewrite (7.20) as

$$\Upsilon^{-1} \Phi_i \hat{\tau}_{ij} \Phi_i^{-1} \Upsilon = \tilde{\tau}_{i\nu_i(j)} = \tilde{\Phi}_i \Upsilon^{-1} \hat{\tau}_{ij} \Upsilon \tilde{\Phi}_i^{-1}.$$

It is equivalent to  $\Upsilon_i^* \hat{\tau}_{ij} = \hat{\tau}_{ij} \Upsilon_i^*$ , where we define

$$\Upsilon_i^* := \Phi_i^{-1} \Upsilon \tilde{\Phi}_i \Upsilon^{-1}.$$

Therefore, by (7.10), in  $\mathcal{C}(\hat{\tau}_i)$ ,  $\tilde{\Phi}_i$  and  $\Phi_i$  are equivalent, if and only if

$$\tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i^* \Upsilon, \quad \Upsilon_i^* \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad i = 1, 2.$$

Conversely, if  $\Upsilon_i^*, \Upsilon$  satisfy (7.17)-(7.19), we take  $\Upsilon_i = \Upsilon_i^* \Upsilon$  to get (7.22) by (7.18). Note that (7.19) ensures that  $\Upsilon$  commutes with  $\hat{\tau}_i$  and  $E_{\hat{\Lambda}_i}$ . Then (7.24), or equivalently (7.23) (i.e. (7.17)) as  $\Upsilon$  commutes with  $E_{\hat{\Lambda}_i}$ , gives us (7.21). By Lemma 7.1, (7.21)-(7.22) are equivalent to (7.20).  $\square$

**Proposition 7.4.** *Let  $\{\tau_{ij}\}$ ,  $\{\tilde{\tau}_{ij}\}$ ,  $\Phi_i$ , and  $\tilde{\Phi}_i$  be as in Proposition 7.3. Decompose  $\Phi_i = \Phi_{i1} \circ \Phi_{i0}^{-1}$  with  $\Phi_{i1} \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{1p})$  and  $\Phi_{i0} \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{1p})$ , and decompose  $\tilde{\Phi}_i$  analogously. Then  $\{\{\tau_{1j}\}, \{\tau_{2j}\}\}$  and  $\{\{\tilde{\tau}_{1j}\}, \{\tilde{\tau}_{2j}\}\}$  are equivalent under a mapping that is tangent to the identity if and only if  $\Phi_{i1} = \tilde{\Phi}_{i1}$  for  $i = 1, 2$ . Assume further that  $\tau_{2j} = \rho\tau_{1j}\rho$  and  $\tilde{\tau}_{2j} = \rho\tilde{\tau}_{1j}\rho$ . Then two families are equivalent under a mapping that is tangent to the identity and commutes with  $\rho$  if and only if  $\Phi_{11} = \tilde{\Phi}_{11}$ .*

*Proof.* When restricting to changes of coordinates that are tangent to the identity, we have  $\Upsilon = I$  in (7.20). Also (7.17) holds trivially as  $\nu_i$  is the identity. By the uniqueness of the decomposition  $\Phi_i = \Phi_{i1}\Phi_{i0}^{-1}$ , (7.18) becomes  $\Phi_{i1} = \tilde{\Phi}_{i1}$ .  $\square$

We consider a general case without restriction on coordinate changes.

**Lemma 7.5.** *Let  $\Upsilon = \text{diag}(\mathbf{a}, \mathbf{a})$  with  $\mathbf{a} \in \{-1, 1\}^p$ . Let  $\mathbf{B}$  be a nonsingular  $p \times p$  matrix and let  $\nu$  be a permutation of  $\{1, \dots, p\}$ . Then*

$$(7.25) \quad \Upsilon^{-1} \circ B_* \circ Z_j \circ B_*^{-1} \circ \Upsilon = B_* \circ Z_{\nu(j)} \circ B_*^{-1}, \quad 1 \leq j \leq p$$

*if and only if*

$$(7.26) \quad \mathbf{B} = (\text{diag } \mathbf{a})^{-1} \mathbf{B} (\text{diag } \mathbf{d}).$$

*In particular, if  $\mathbf{B}$  is an upper or lower triangular matrix, then  $\nu = I$  and  $\mathbf{d} = \mathbf{a}$ .*

*Proof.* Let  $\tilde{\mathbf{Z}}_j = \text{diag}(1, \dots, -1, \dots, 1)$  be the matrix where  $-1$  at the  $j$ -th place. Set  $\mathbf{C} := \mathbf{B}^{-1} \text{diag } \mathbf{a} \mathbf{B}$  and  $\mathbf{C} = (c_{ij})$ . In  $2 \times 2$  block matrices, we see that (7.25) is equivalent to  $\mathbf{C} \tilde{\mathbf{Z}}_{\nu(j)} = \tilde{\mathbf{Z}}_j \mathbf{C}$ , i.e.

$$-c_{i\nu(j)} = c_{ij}, \quad i \neq j.$$

Therefore,  $\mathbf{C} = \text{diag } \mathbf{d}$  with  $d_j = c_{j\nu(j)}$ , by (3.20).  $\square$

We will assume that  $M$  is a higher order perturbation of non-resonant product quadric. Let us recall  $\hat{\sigma}$  be given by (7.3) and define  $\hat{\tau}_{ij}$  as follows:

$$\hat{\sigma} : \begin{cases} \xi'_j = \hat{M}_j(\xi\eta)\xi_j \\ \eta'_j = \hat{M}_j^{-1}(\xi\eta)\eta_j, \end{cases} \quad \hat{\tau}_{ij} : \begin{cases} \xi'_j = \hat{\Lambda}_{ij}(\xi\eta)\eta_j \\ \eta'_j = \hat{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j \\ \xi'_k = \xi_k \\ \eta'_k = \eta_k, \quad k \neq j \end{cases}$$

with  $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$  and  $\hat{M}_j = \hat{\Lambda}_{1j}^2$ . Let  $\hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{1p}$ . Recall that  $E_{\hat{\Lambda}_i}$  in (7.5).

**Proposition 7.6.** *Let  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  be the family of involutions with  $\rho$  be given by (1.3). Suppose that the linear parts of  $\tau_{1j}$  are given by (7.6) and associated  $\sigma$  is non-resonant, while the associated matrix  $\mathbf{B}$  for  $\{T_{1j}\}$  satisfies the non-degeneracy condition that (7.26) holds only for  $\nu = I$ . Let  $\hat{\sigma}$  be the formal normal form  $\hat{\sigma}$  of the  $\sigma$  associated to  $M$  that is given by (7.3) in which  $\log \hat{M}$  is in the formal normal form (5.35). Let  $\hat{\tau}_{1j}$  be given by (7.9) and  $\hat{\tau}_{2j} = \rho\hat{\tau}_{1j}\rho$ . In suitable formal coordinates the involutions  $\tau_{ij}$  have the form*

$$(7.27) \quad \tau_{1j} = \Psi \hat{\tau}_{1j} \Psi^{-1}, \quad \tau_{2j} = \rho \tau_{1j} \rho, \quad \Psi \in \mathcal{C}(\hat{\tau}_1) \cap C^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}), \quad \Psi'(0) = \mathbf{I}.$$

Moreover, if  $\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}$  have the form (7.27) in which  $\Psi$  is replaced by  $\tilde{\Psi}$ . Then there exists a formal mapping  $R$  commuting with  $\rho$  and transforms the family  $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}\}$  into  $\{\tau_{11}, \dots, \tau_{1p}\}$  if and only if  $R$  is an  $R_\epsilon$  defined by (7.1) and

$$(7.28) \quad \tilde{\Psi} = R_\epsilon^{-1} \Psi R_\epsilon, \quad R_\epsilon \rho = \rho R_\epsilon.$$

In particular,  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  is formally equivalent to  $\{\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}, \rho\}$  if and only if  $\Psi$  in (7.27) is the identity map.

*Proof.* We apply Proposition 7.3. We need to refine the equivalence relation (7.17)-(7.19). First we know that (7.19) means that  $\Upsilon = R_\epsilon$  and it commutes with  $\rho$ . It remains to refine (7.18). We have  $\Phi_2 = \rho \Phi_1 \rho$ . By assumption, we know that  $\nu_1$  in (7.17) must be the identity. Then  $\Phi_1 \in \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$  implies that  $\Upsilon^{-1} \Phi_1 \Upsilon \in \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ ; indeed by (7.26) we have

$$\Upsilon E_{\Lambda_i} \circ (B_i)_* = E_{\Lambda_i} \circ \Upsilon \circ (B_i)_* = E_{\Lambda_i} \circ (B_i)_* \circ D, \quad \tilde{\mathbf{D}} = \text{diag}(\text{diag } \mathbf{a}, \text{diag } \mathbf{d}).$$

Note that  $\psi_0 = (U, V)$  is in  $\mathcal{C}_2(Z_1, \dots, Z_p)$  if and only if

$$U(\xi, \eta) = \tilde{U}(\xi, \eta_1^2, \dots, \eta_p^2), \quad V_j(\xi, \eta) = \eta_j \tilde{V}_j(\xi, \eta_1^2, \dots, \eta_p^2).$$

Let  $\psi_1 = (U, V)$  be in  $\mathcal{C}_2^c(Z_1, \dots, Z_p)$ , i.e.

$$U(\xi, \eta) = \sum_i \eta_i \tilde{U}_i(\xi, \eta_1^2, \dots, \eta_i^2), \quad V_j(\xi, \eta) = V_j^*(\xi, \eta) + \eta_j \sum_i \eta_i \tilde{V}_i(\xi, \eta_1^2, \dots, \eta_i^2),$$

where  $V_j^*(\xi, \eta)$  is independent of  $\eta_j$ . Since  $\mathbf{D}$  is diagonal, then  $D\psi_1 D^{-1}$  is in  $\mathcal{C}_2^c(Z_1, \dots, Z_p)$ . This shows that conjugation by  $\Upsilon$  preserves  $\mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ . Also  $\Upsilon$  commutes with each  $\hat{\tau}_{1j}$ . Hence, it preserves  $\mathcal{C}(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ . By the uniqueness of decomposition, (7.18) becomes

$$\tilde{\Phi}_{11} = \Upsilon^{-1} \Phi_{11} \Upsilon, \quad \tilde{\Phi}_{10}^{-1} = \Upsilon^{-1} \Phi_{10}^{-1} \Upsilon^* \Upsilon.$$

The second equation defines  $\Upsilon_1^*$  that is in  $\mathcal{C}(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$  as  $\Upsilon, \Phi_{10}, \tilde{\Phi}_{10}$  are in the centralizer. Rename  $\Phi_{11}, \tilde{\Phi}_{11}$  by  $\Psi, \tilde{\Psi}$ . This shows that the equivalence relation is reduced to (7.28).  $\square$

We now derive the following formal normal form.

**Theorem 7.7.** *Let  $M$  be a real analytic submanifold that is a higher order perturbation of a non-resonant product quadric. Assume that the formal normal form  $\hat{\sigma}$  of the  $\sigma$  associated to  $M$  is given by (7.3) in which  $\log \hat{M}$  is tangent to the identity and in the formal normal form (5.35). Let  $E_{\hat{\Lambda}_1}$  be defined by (7.4). Then  $M$  is formally equivalent to a formal submanifold in the  $(z_1, \dots, z_{2p})$ -space defined by*

$$\tilde{M}: z_{p+j} = (\lambda_j^{-1} U_j(\xi, \eta) - V_j(\xi, \eta))^2, \quad 1 \leq j \leq p,$$

where  $(U, V) = E_{\hat{\Lambda}_1(0)} E_{\hat{\Lambda}_1}^{-1} \Psi^{-1}$ ,  $\Psi$  is tangent to the identity and in  $\mathcal{C}(\hat{\tau}_1) \cap \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ , defined in Lemma 7.2, and  $\xi, \eta$  are solutions to

$$z_j = U_j(\xi, \eta) + \lambda_j V_j(\xi, \eta), \quad \bar{z}_j = \overline{U_j \circ \rho(\xi, \eta)} + \overline{\lambda_j V_j \circ \rho(\xi, \eta)}, \quad 1 \leq j \leq p.$$

Furthermore, the  $\Psi$  is uniquely determined up to conjugacy  $R_\epsilon \Psi R_\epsilon^{-1}$  by an involution  $R_\epsilon: \xi_j \rightarrow \epsilon_j \xi_j, \eta_j \rightarrow \epsilon_j \eta_j$  for  $1 \leq j \leq p$  that commutes with  $\rho$ , i.e.  $\epsilon_{s+s_*} = \epsilon_s$ . The formal holomorphic automorphism group of  $\hat{M}$  consists of involutions of the form

$$L_\epsilon: z_j \rightarrow \epsilon_j z_j, \quad z_{p+j} \rightarrow z_{p+j}, \quad 1 \leq j \leq p$$



with  $\epsilon$  satisfying  $R_\epsilon \Psi = \Psi R_\epsilon$  and  $\epsilon_{s+s^*} = \epsilon_s$ . If the  $\sigma$  associated to  $M$  is holomorphically equivalent to a Poincaré-Dulac normal form, then  $\tilde{M}$  can be achieved by a holomorphic transformation too.

*Proof.* We first choose linear coordinates so that the linear parts of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$  are in the normal form in Lemma 3.2. We apply Proposition 7.6 and assume that  $\tau_{ij}$  are already in the normal form. The rest of proof is essentially in Proposition 2.1 and we will be brief. Write  $T_{1j} = E_{\hat{\Lambda}_1(0)} \circ Z_j \circ E_{\hat{\Lambda}_1(0)}^{-1}$ . Let  $\psi = (U, V)$  with  $U, V$  being given in the theorem. We obtain

$$\tau_{1j} = \Psi \hat{\tau}_{1j} \Psi^{-1} = \psi^{-1} T_{1j} \psi, \quad 1 \leq j \leq p.$$

Let  $f_j = \xi_j + \lambda_j \eta_j$  and  $h_j = (\lambda_j \xi_j - \eta_j)^2$ . The invariant functions of  $\{T_{11}, \dots, T_{1p}\}$  are generated by  $f_1, \dots, f_p, h_1, \dots, h_p$ . This shows that the invariant functions of  $\{\tau_{11}, \dots, \tau_{1p}\}$  are generated by  $f_1 \circ \psi, \dots, f_p \circ \psi, h_1 \circ \psi, \dots, h_p \circ \psi$ . Set  $g := \overline{f \circ \psi \circ \rho}$ . We can verify that  $\phi = (f \circ \psi, g)$  is biholomorphic. Now  $\phi \rho \phi^{-1} = \rho_0$ . Let  $M$  be defined by

$$z_{p+j} = E_j(z', \bar{z}'), \quad 1 \leq j \leq p,$$

where  $E_j = h_j \circ \phi^{-1}$ . Then  $E_j \circ \phi$  and  $z_j \circ \phi = f_j$  are invariant by  $\{\tau_{1k}\}$ . This shows that  $\{\phi \tau_{ij} \phi^{-1}\}$  has the same invariant functions as deck transformations of  $\pi_1$  of the complexification  $\mathcal{M}$  of  $M$ . By Lemma 2.5 in [GS15],  $\{\phi \tau_{1j} \phi^{-1}\}$  agrees with the unique set of generators for the deck transformations of  $\pi_1$ . Then  $M$  is a realization of  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ .

Finally, we identify the formal automorphisms of  $M$ , which fix the origin. For such an automorphism  $F$  on  $\mathbf{C}^n$ , define  $\tilde{F}(z', w') = (F(z', E(z', w')), \overline{F}(w', \overline{E}(w', z')))$  on  $\mathcal{M}$ . Then  $\phi^{-1} \tilde{F} \phi$  preserves  $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ . By Proposition 7.6,  $\phi^{-1} \tilde{F} \phi = R_\epsilon$ ,  $R_\epsilon \rho = \rho R_\epsilon$ , and  $R_\epsilon \Psi = \Psi R_\epsilon$ . Given (7.1), we write  $R_\epsilon = (L'_\epsilon, L'_\epsilon)$ . In view of  $(U, V) = E_{\hat{\Lambda}_1(0)} E_{\hat{\Lambda}_1}^{-1} \Psi^{-1}$ , we obtain that  $L'_\epsilon U = U R_\epsilon$  and  $L'_\epsilon V = V R_\epsilon$ . Since  $z_j = U_j(\xi, \eta) + \lambda_j V_j(\xi, \eta)$  and  $z_{p+j} = (\lambda_j^{-1} U_j(\xi, \eta) - V_j(\xi, \eta))^2$ , then  $z' \circ \tilde{F} = L'_\epsilon z'$  and  $z'' \circ \tilde{F} = z''$  as functions in  $(z', w')$ . This shows that  $z' \circ F = L'_\epsilon z'$  and  $z'' \circ F = z''$  as functions in  $(z', z'')$ . Therefore,  $F = L_\epsilon$ .  $\square$

**Remark 7.8.** Let  $b$  be on the unit circle with  $0 \leq \arg b < \pi$ . Let

$$\mathbf{B} = \begin{pmatrix} 1 & b \\ \tilde{b} & 1 \end{pmatrix}, \quad |\tilde{b}| \leq 1, \quad b\tilde{b} \neq 1.$$

One can check that (7.26) admits a solution  $\nu \neq I$  if and only if  $\tilde{b} = -\bar{b}$ .

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